THE THEORY OF THE
RIEMANN
ZETA-FUNCTION

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PREFACE TO THE SECOND EDITION

Since the first edition was written, a vast amount of further work has been done. This has been covered by the end of chapter notes. In most instances, restrictions on space have prohibited the inclusion of full proofs, but I have tried to give an indication of the methods used wherever possible. (Proofs of quite a few of the recent results described in the end of chapter notes may be found in the book by Ivic [1].) I have also corrected a number of minor errors, and made a few other small improvements to the text. A considerable number of recent references have been added.

In rewriting this work I have had help from Professors J. B. Conrey, P. D. T. A. Elliott, A. Ghosh, S. M. Gonek, H. L. Montgomery, and S. J. Patterson. It is a pleasure to record my thanks to them.
PREFACE TO FIRST EDITION

This book is a successor to my Cambridge Tract *The Zeta-Function of Riemann*, 1930, which is now out of print and out of date. It seems no longer practicable to give an account of the subject in such a small space as a Cambridge Tract, so that the present work, though on exactly the same lines as the previous one, is on a much larger scale. As before, I do not discuss general prime-number theory, though it has been convenient to include some theorems on primes.

Most of this book was compiled in the 1890's, when I was still researching on the subject. It has been brought partly up to date by including some of the work of A. Selberg and of Vinogradov, though a great deal of recent work is scantily represented.

The manuscript has been read by Dr. S. H. Min and by Prof. D. B. Sears, and my best thanks are due to them for correcting a large number of mistakes. I must also thank Prof. F. V. Atkinson and Dr. T. M. Fleet for their kind assistance in reading the proof-sheets.

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Since all integers up to $P$ are of this form, it follows that, if $\zeta(s)$ is defined by (1.1.1),

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \left(\zeta(s) - 1 - \frac{1}{m_1} - \frac{1}{m_2} - \cdots\right) \leq \frac{1}{P^{(s+1)/2}} \left(1 + \frac{1}{P^{s+1}}\right).$$

This tends to 0 as $s \to \infty$, if $s > 1$; and (1.1.2) follows.

This fundamental identity is due to Euler, and (1.1.2) is known as Euler's product. But Euler considered it for particular values of $s$ only, and it was Riemann who first considered $\zeta(s)$ as an analytic function of a complex variable.

Since a convergent product of non-zero factors is not zero, we deduce that $\zeta(s)$ has no zeros for $s > 1$. This may be proved directly as follows. For $s > 1$

$$\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^2}\right) \cdots \left(1 - \frac{1}{p^m}\right) \cdots = 1 + \frac{1}{m_1} + \frac{1}{m_2} + \cdots,$$

where $m_1, m_2, \ldots$ are the integers all of whose prime factors exceed $P$. Hence

$$\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^2}\right) \cdots \left(1 - \frac{1}{p^m}\right) \cdots \geq 1 - \frac{1}{(P+1)^{s-1}} - \frac{1}{(P+2)^{s-1}} - \cdots > 0$$

if $P$ is large enough. Hence $\zeta(s) > 0$.

The importance of $\zeta(s)$ in the theory of prime numbers lies in the fact that it contains two expressions, one of which contains the primes explicitly, while the other does not. The theory of primes is largely concerned with the function $\sigma(x)$, the number of primes not exceeding $x$. We can transform (1.1.2) into a relation between $\zeta(s)$ and $\sigma(x)$; for if $s > 1$,

$$\log \zeta(s) = -\sum_{n=2} \frac{\sigma(n)}{n^s} = \sum_{n=2} \frac{\pi(n) - \pi(n-1)}{n^s} \log\left(1 - \frac{1}{n^s}\right)$$

$$= -\sum_{n=2} \frac{\pi(n)}{n^s} \log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{P^{s+1}}\right)$$

$$= -\sum_{n=2} \frac{\pi(n)}{n^s} \frac{1}{x(x-1)} dx = \sum_{n=2} \frac{\pi(x)}{x(x-1)} dx.$$

(1.1.3)

The rearrangement of the series is justified since $\pi(n) \ll n$ and

$$\log(1-n^{-s}) = o(n^{-s}).$$

Again,

$$\frac{1}{\zeta(s)} = \prod_{p} \left(1 - \frac{1}{p^s}\right),$$

and on carrying out the multiplication we obtain

$$\frac{1}{\zeta(s)} = \sum_{n=2} \frac{\mu(n)}{n^s} \quad (s > 1),$$

(1.1.4)

where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n$ is the product of $k$ different primes, and $\mu(n) = 0$ if $n$ contains any factor to a power higher than the first. The process is easily justified as in the case of $\zeta(s)$.

The function $\mu(n)$ is known as the Möbius function. It has the property

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & (n = 1), \\ 0 & (n > 1). \end{cases}$$

(1.1.5)

where $d|n$ means that $d$ is a divisor of $n$. This follows from the identity

$$1 = \sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{\mu(d)}{d^s} = \sum_{d|n} \frac{1}{d^s} \sum_{d|n} \mu(d).$$

It also gives the Möbius inversion formula

$$g(n) = \sum_{d|n} f(d),$$

(1.1.6)

$$f(n) = \sum_{d|n} \mu(n/d) g(d),$$

(1.1.7)

connecting two functions $f(n), g(n)$ defined for integral $n$. If $f$ is given and $g$ defined by (1.1.6), the right-hand side of (1.1.7) is

$$\sum_{d|n} \mu(n/d) \sum_{d|n} f(d).$$

The coefficient of $f(n)$ is $\mu(n)$ if $s = 1$. If $t < q$, then $d = kr$, where $k|q/r$. Hence the coefficient of $f(r)$ is

$$\sum_{k|q/r} \mu(k/r) = \sum_{k|q} \mu(k) = 0$$

by (1.1.6). This proves (1.1.7). Conversely, if $g$ is given, and $f$ is defined by (1.1.7), then the right-hand side of (1.1.6) is

$$\sum_{d|n} \mu(n/d) f(d),$$

and this is $g(n)$, by a similar argument. The formula may also be
derived formally from the obviously equivalent relations

\[ F(s) = \sum_{n=1}^{\infty} \frac{\mathcal{g}(n)}{n^s}, \quad F(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\mathcal{g}(n)}{n^s}, \]

where

\[ F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \]

Again, on taking logarithms and differentiating (1.1.2), we obtain, for \( s > 1 \),

\[
\frac{\zeta(s)}{\zeta(2s)} = -\sum_{p} \log p \frac{\zeta(p)}{p^s} \left( \frac{1}{p^s} \right)^2 = -\sum_{p} \log p \sum_{k=1}^{\infty} \frac{1}{p^{sk}} = -\sum_{k=1}^{\infty} \Lambda(n) \frac{1}{n^s}, \tag{1.1.8}
\]

where \( \Lambda(n) = \log p \) if \( n = p \) or a power of \( p \), and otherwise \( \Lambda(n) = 0 \). On integrating we obtain

\[ \log \zeta(s) = -\sum_{k=1}^{\infty} \Lambda(n) \frac{1}{n^s} \quad (s > 1), \tag{1.1.9}
\]

where \( \Lambda(n) = \Lambda(n)/\log n \), and the value of \( \log \zeta(s) \) is that which tends to 0 as \( s \to \infty \), for any fixed \( t \).

### 1.2. Various Dirichlet series connected with \( \zeta(s) \).

In the first place

\[ \mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (s > 1), \tag{1.2.1}
\]

where \( d(n) \) denotes the number of divisors of \( n \) (including 1 and \( n \) itself). For

\[ \mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} 1 = \sum_{d=1}^{\infty} \frac{1}{d^s} \sum_{n=1}^{\infty} 1,
\]

and the number of terms in the last sum is \( d(n) \). And generally

\[ \mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (s > 1), \tag{1.2.2}
\]

where \( k = 2, 3, 4, \ldots \), and \( d_k(n) \) denotes the number of ways of expressing \( n \) as a product of \( k \) factors, expressions with the same factors in a different order being counted as different. For

\[ \mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \frac{1}{d^s} \sum_{e|d} 1,
\]

and the last sum is \( d_k(n) \).

### 1.3. The Dirichlet series

Since we have also

\[ \mathcal{D}(s) = \prod_{p} \left( 1 - \frac{1}{p^s} \right) = \prod_{p} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots \right), \tag{1.2.3}
\]

on comparing the coefficients in (1.2.1) and (1.2.3) we verify the elementary formula

\[ d(n) = (m_1+1) \ldots (m_r+1) \tag{1.2.4}
\]

for the number of divisors of

\[ n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}. \tag{1.2.5}
\]

Similarly from (1.2.2)

\[ d_k(n) = (k+m-1)! (k+m-1)! \ldots (k+m-1)! \tag{1.2.6}
\]

We next note the expansions

\[ \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (s > 1), \tag{1.2.7}
\]

where \( d(n) \) is the coefficient in (1.1.4),

\[ \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (s > 1), \tag{1.2.8}
\]

where \( v(n) \) is the number of different prime factors of \( n \);

\[ \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\zeta(s)}{n^s} \quad (s > 1). \tag{1.2.10}
\]

To prove (1.2.7), we have

\[ \frac{\zeta(s)}{\zeta(2s)} = \prod_{p} \left( 1 - \frac{1}{p^s} \right) = \prod_{p} \left( 1 + \frac{1}{p-2} \right), \tag{1.2.9}
\]

and this differs from the formula for \( 1/\zeta(s) \) only in the fact that the signs are all positive. The result is therefore clear. To prove (1.2.8), we have

\[ \frac{\zeta(s)}{\zeta(2s)} = \prod_{p} \left( 1 + \frac{1}{p^s} \right) = \prod_{p} \left( 1 - \frac{1}{p^s} \right) = \prod_{p} \left( 1 + \frac{1}{p-2} \right), \tag{1.2.10}
\]
and the result follows. To prove (1.2.0),
\[
\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \prod_p \frac{1 + p^{-s}}{(1 - p^{-s})^2} = \prod_p \left( \frac{(1 + p^{-s})(1 + 2p^{-s} + 3p^{-2s} + \ldots)}{(1 - p^{-s})^2} \right)
\]
and the result follows, since, if \( n \) is (1.2.2),
\[
d(n) = (2m+1)_n \cdot \ldots \cdot (2m+1).
\]
Similarly
\[
\zeta(s) = \prod_p \frac{1 - p^{-2s}}{(1 - p^{-s})^2} = \prod_p \frac{1 + p^{-s}}{(1 - p^{-s})^2} = \prod_p \left( \frac{(1 + p^{-s})(1 + 3p^{-2s} + \ldots + (m+1)(m+2)p^{-ms} + \ldots)}{(1 - p^{-s})^2} \right)
\]
and the result follows, since (1.2.10).

Other formulae are
\[
\zeta(2s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad (s > 1),
\]
where \( \lambda(n) = (-1)^k \) if \( n \) has \( k \) prime factors, one factor of degree \( k \) being counted \( k \) times;
\[
\zeta(2k) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2k}} \quad (s > 1),
\]
where \( \phi(n) \) is the number of numbers less than \( n \) and prime to \( n \); and
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (s > 2),
\]
where \( a(n) \) is the greatest odd divisor of \( n \). Of these, (1.2.11) follows at once from
\[
\frac{\zeta(2s)}{\zeta(s)} = \prod_p \left( \frac{1 - p^{-2s}}{(1 - p^{-s})^2} \right) = \prod_p \left( \frac{1 + p^{-s}}{(1 - p^{-s})^2} \right) = \prod_p \left( \frac{1 + p^{-s}}{(1 - p^{-s})^2} \right)
\]
also
\[
\frac{\zeta(s-1)}{\zeta(s)} = \prod_p \left( \frac{1 - p^{-s}}{(1 - p^{-s})^2} \right) = \prod_p \left( \frac{1 - p^{-s}}{(1 - p^{-s})^2} \right).
\]

and (1.2.12) follows, since, if \( n \to \rho_1 \rho_2 \ldots \rho_s \),
\[
\phi(n) = n \left( 1 - \frac{1}{\rho_1} \right) \left( 1 - \frac{1}{\rho_2} \right) \ldots
\]
Finally
\[
\frac{1 - 2^{-s}}{1 - 2^{-s}} \zeta(s-1) = \frac{1 - 2^{-s}}{1 - 2^{-s}} \prod_p \frac{1}{1 - p^{-s}}
\]
\[
= \prod_p \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots \right) \left( 1 + \frac{3}{p^3} + \frac{3}{p^5} + \ldots \right)
\]
and (1.2.13) follows.

Many of these formulae are, of course, simply particular cases of the general formula
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots \right),
\]
where \( f(n) \) is a multiplicative function, i.e., is such that, if \( n = \rho_1 \rho_2 \ldots \rho_s \ldots \), then
\[
f(n) = f(\rho_1)f(\rho_2)f(\rho_3)\ldots
\]
Again, let \( \lambda(n) \) denote the number of representations of \( n \) as a product of \( k \) factors, each greater than unity when \( n > 1 \), the order of the factors being essential. Then clearly
\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = (\zeta(s)-1)^k \quad (s > 1).
\]
Let \( f(n) \) be the number of representations of \( n \) as a product of \( k \) factors greater than unity, representations with factors in a different order being considered as distinct; and let \( f(1) = 1 \). Then
\[
f(n) = \sum_{\lambda(n)} f(n).
\]
Hence
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = 1 + \sum_{\lambda(n)} \frac{\lambda(n)-1}{1 - \lambda(n)} = \frac{\zeta(s)-1}{\zeta(s)}
\]
(1.2.15)

It is easily seen that \( \zeta(s) = \frac{2}{a} \) for \( \sigma = a \), where \( a \) is a real number greater than 1; and \( \zeta(s) < \frac{2}{a} \) for \( \sigma > a \), so that (1.2.10) holds for \( \sigma > a \).
1.3. Sums involving \( \sigma_a(n) \). Let \( \sigma_a(n) \) denote the sum of the \( a \)th powers of the divisors of \( n \). Then

\[
\zeta(a)(t-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^t} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^t} = \sum_{n=1}^{\infty} \frac{1}{n^t} \sum_{d \mid n} d^a,
\]

i.e.

\[
\zeta(a)(t-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^t} \quad (a > 1, \sigma > \Re(a)+1).
\] (1.3.1)

Since the left-hand side is, if \( a \neq 0 \),

\[
\prod_p \left(1 + \frac{1}{p^t + \frac{\beta}{p^t} - \frac{\gamma}{p^t} + \cdots} \right) = \prod_p \left(1 + \frac{1}{p^t + \frac{\beta}{p^t} + \cdots} \right) = \prod_p \left(1 + \frac{1}{p^t + \frac{\beta}{p^t}} \right)
\]

we have

\[
\sigma_a(n) = \frac{1-p^{a-1}}{1-p^t}, \quad 1-p^{a-1} \quad (1-p^t).
\] (1.3.2)

if \( n \) is (1.2.5), as is also obvious from elementary considerations.

The formula

\[
\zeta(a)(t-a) \zeta(t-b) \zeta(t-a-b) = \sum_{n=1}^{\infty} \frac{\sigma_a(n) \sigma_b(n)}{n^t}
\] (1.3.3)

is valid for \( a > \max(1, \Re(a)+1, \Re(b)+1, \Re(a+b)+1) \). The left-hand side is equal to

\[
\prod_p \left(1-p^{a-1}(1-p^{b-1})(1-p^{a+b-2}) \right)
\]

Putting \( p^{-t} = n \), the partial-fraction formula gives

\[
1-p^{a+b-1} = (1-n)(1-p^{a-2})(1-p^{b-2})(1-p^{a+b-2})
\]

Hence

\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^t} = \prod_{p} \sum_{\sigma=0}^{\infty} \frac{\sigma_a(p^\sigma)}{p^{t\sigma}}.
\]

Now if \( a 
eq 0 \),

\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^t} = \zeta(t-a) \prod_{p} \left(1 + \frac{\sigma_a(p^\sigma)}{p^{t\sigma}} \right) \]

Making \( a = 0 \),

\[
\sum_{n=1}^{\infty} \frac{d(n)}{n^t} = \zeta(t) \prod_{p} \left(1 + \frac{p^{t-1}}{1-p^t} \right).
\] (1.4.2)
1.5. Ramanujan's sums.† Let
\[ c_k(n) = \sum_{d|n} \mu(\frac{n}{d}) \delta_{k}(d), \tag{1.5.1} \]
where \( k \) runs through all positive integers less than and prime to \( k \). Many formulae involving these sums were proved by Ramanujan.

We shall first prove that
\[ c_k(n) = \sum_{d|n} \mu(d) \delta_{k}(\frac{n}{d}). \tag{1.5.2} \]
The sum
\[ \eta_k(n) = \sum_{m|n} \frac{1}{\phi(m)} \]
is equal to \( k \) if \( k \) is a prime and 0 otherwise. Denoting by \((r, d)\) the highest common factor of \( r \) and \( d \), so that \((r, d) = 1\) means that \( r \) is prime to \( d \),
\[ \delta_k(n) = \sum_{r=d}^{k-1} \sum_{d|n} \delta_{(r,d)}(d) e^{-\pi i \frac{r}{k} \frac{n}{d}}, \eta_k(n) = \eta_k(d). \]
Hence by the inversion formula of Möbius (1.1.7)
\[ c_k(n) = \sum_{\sigma|n} \mu(\sigma) \eta_k(n), \]
and (1.5.2) follows. In particular
\[ c_k(1) = \mu(k). \tag{1.5.3} \]
The result can also be written
\[ c_k(n) = \sum_{d|n} \mu(d) \delta_{k}(\frac{n}{d}). \]
Hence
\[ \sum_k c_k(n) = \sum_{d|n} \mu(d) \delta_{k}(\frac{n}{d}) = \sum_{d|n} \mu(d) \delta_{k}(\frac{n}{d}) \tag{1.5.4} \]
the series being absolutely convergent for \( \sigma > 1 \) since \( |c_k(n)| \leq c_k(n) \), by (1.5.3). We have also
\[ \sum_{k=1}^{\infty} c_k(n) = \sum_{d|n} \sum_{k=1}^{\infty} \mu(\frac{n}{d}) \delta_{k}(d) \]
\[ = \sum_{d|n} \mu(\frac{n}{d}) \delta_{k}(d) - \sum_{d|n} \frac{1}{\phi(d)} = \zeta(\sigma) \sum_{d|n} \mu(\frac{n}{d}) d^{1-\sigma}. \tag{1.5.5} \]

† Ramanujan (3), Hardy (5).
‡ Two more proofs are given by Hardy, Ramanujan, 137-41.

1.8 THE DIHORICLIT SERIES

We can also sum series of the form†
\[ \sum_{n=1}^{\infty} c_k(n) \delta_k(n), \]
where \( f(n) \) is a multiplicative function. For example,
\[ \sum_{n=1}^{\infty} c_k(n) \delta_k(n) \]
\[ = \sum_{n=1}^{\infty} \frac{d(n)}{n^\sigma} \sum_{d|n} \delta_k(d) \]
\[ = \zeta(\sigma) \sum_{\delta|n} \delta^{1-\sigma} \zeta(\sigma) \prod_{p|\delta} \left( 1 - \frac{1}{p-\sigma} \right). \tag{1.5.6} \]

Hence
\[ \sum_{n=1}^{\infty} \frac{c_k(n) \delta_k(n)}{n^\sigma} = \zeta(\sigma) \delta(\sigma) \prod_{p|\delta} \left( 1 - \frac{1}{p-\sigma} \right). \tag{1.5.6} \]

For example, in the simplest case \( f(n) = 1 \), the series is
\[ \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \sum_{\delta|n} \delta^{1-\sigma} \]
for given \( \delta \), runs through those multiples of \( \delta \) which are integers.

If \( \delta \) in its lowest terms is \( \delta/\delta \), these are the numbers \( \delta, 2\delta, \ldots \).

Hence the sum is
\[ \sum_{\delta|n} \delta^{1-\sigma} \sum_{\delta|n} \delta^{1-\sigma} \delta^{1-\sigma}, \]
† Gross (1).
Since $\delta_1 = \delta/(q, b)$, the result is
\[
\sum_{n=1}^{\infty} \frac{\delta(q, n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} 3^{s-n} \delta(q, b)^n.
\]  
(5.1.7)

1.6. There is another class of identities involving infinite series of eta-functions. The simplest of these is
\[
\sum_{n=1}^{\infty} \frac{1}{p^n} = \sum_{n=1}^{\infty} \frac{\mu(n) \log \zeta(ns)}{n},
\]
(6.1.1)

where $P(n) = \sum p^{-n}$. Hence
\[
\sum_{n=1}^{\infty} \frac{\mu(n) \log \zeta(ns)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n) \log \zeta(n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n) \log \zeta(n)}{n},
\]
and the result follows from (1.1.5).

A closely related formula is
\[
\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n) \log \zeta(ns)}{n},
\]  
(6.1.2)

where $s(n)$ is defined under (1.2.8). This follows at once from (6.1.1) and the identity
\[
\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{p|n} 1 = \frac{1}{p^s}.
\]

Denoting by $b(n)$ the number of divisors of $n$ which are prime or powers of primes, another identity of the same class is
\[
\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\phi(n) \log \zeta(ns)}{n},
\]  
(6.1.3)

where $b(n)$ is defined under (1.2.12). For the left-hand side is equal to
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{p|n} \frac{1}{p^s} \sum_{p|n} \frac{1}{p^s} + \ldots,
\]
and the series on the right is
\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{p|n} \frac{1}{p^s} \sum_{p|n} \frac{1}{p^s} \phi(n).
\]
Since the result follows.

\footnote{See Landau and Wallis (1), Edwardsen (1), (2).}

II
THE ANALYTIC CHARACTER OF $\zeta(s)$, AND THE FUNCTIONAL EQUATION

2.1. Analytic continuation and the functional equation, first method. Each of the formulæ of Chapter I is proved on the supposition that the series or product concerned is absolutely convergent. In each case this restricts the region where the formulæ is proved to be valid to a half-plane. For $\zeta(s)$ itself, and in all the fundamental formulæ of § 1.1, this is the half-plane $s > 1$.

We have next to inquire whether the analytic function $\zeta(s)$ can be continued beyond this region. The result is

\textbf{Theorem 2.1.} The function $\zeta(s)$ is regular for all values of $s$ except $s = 1$, where there is a simple pole with residue 1. It satisfies the functional equation
\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s).
\]  
(2.1.1)

This can be proved in a considerable variety of different ways, some of which will be given in later sections. We shall first give a proof depending on the following summation formulæ.

Let $\phi(x)$ be any function with a continuous derivative in the interval $[a, b]$. Then, if $[x]$ denotes the greatest integer not exceeding $x$,
\[
\sum_{x \leq \alpha} \phi(x) = \int_a^b \phi(x) \, dx + \int_a^b \left( [x] - x \right) \phi'(x) \, dx + (b - [b] - 1)\phi(b) + (a - [a] - 1)\phi(a).
\]  
(2.1.2)

Since the formulæ is plainly additive with respect to the interval $[a, b]$, it suffices to suppose that $n \subset a < b \subset n + 1$. One then has
\[
\int_a^b (x + \frac{1}{2}) \phi(x) \, dx = \int_a^b \phi(x) \, dx + \int_a^b (x + \frac{1}{2}) \phi'(x) \, dx + \phi(a)\frac{3}{2}.
\]  
(2.1.3)
on integrating by parts. Thus the right hand side of (2.1.2) reduces to
\((\frac{1}{a}) - \ln d\ell d\theta\). This vanishes unless \(h = n = 1\), in which case it is
\(n\frac{1}{a} + 2\), as required.

In particular, let \(d\ell = n\pi\), where \(s \neq 1\), and let \(a\) and \(\beta\) be positive integers. Then
\[
\sum_{s=1}^{\infty} \frac{1}{s^n} = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} - \frac{\beta^n - \alpha^n}{\beta - \alpha} - \frac{1}{s^n} + \frac{1}{s^{n+1} + \beta^n - \alpha^n}.
\]  

(2.1.3)

First take \(s > 1\), \(a = 1\), and make \(\beta \to \infty\). Adding 1 to each side, we obtain
\[
\theta(s) = \int \frac{[s] - \frac{1}{s}}{\beta^{n+1}}\,ds + \frac{1}{s^{n+1} - \beta^n} + \frac{1}{\beta^{n+1} + \beta^n - \alpha^n}.
\]  

(2.1.4)

Since \([s] - \frac{1}{s}\) is bounded, this integral is convergent for \(s > 0\), and
uniformly convergent in any finite region to the right of \(a = 0\). It therefore
defines an analytic function of \(s\), regular for \(a > 0\). The
right-hand side therefore provides the analytic continuation of \(\theta(s)\) up to
\(s = 0\), and there is clearly a simple pole at \(s = 1\) with residue 1.

For \(0 < s < 1\) we have
\[
\int_{s}^{1} \frac{[s] - \frac{1}{s}}{\beta^{n+1}}\,dx = \int_{1}^{\infty} \frac{x^n - s^n}{\beta^{n+1}}\,dx = \frac{1}{s^{n+1}} - \frac{1}{s^{n+1}} - \frac{1}{\beta^n} = \frac{s^n}{\beta^n}.
\]

(2.1.5)

and (2.1.4) may be written
\[
\theta(s) = \int_{s}^{1} \frac{[s] - \frac{1}{s}}{\beta^{n+1}}\,dx \quad (0 < s < 1).
\]

Actually (2.1.4) gives the analytic continuation of \(\theta(s)\) for \(s > -1\); for \(f(s) = [s] - \frac{1}{s}\), 
\(f(s) = \int f(s)\,ds\),

then \(f(s)\) is also bounded, since, as is easily seen, 
\[
\int_{a}^{b} f(y)\,dy = 0
\]

for any integer \(k\). Hence
\[
\int_{1}^{\infty} \frac{[x] - \frac{1}{x}}{\beta^{n+1}}\,dx = \left[\frac{[x]}{\beta^{n+1}}\right]_{1}^{\infty} + \frac{1}{s^{n+1} - \beta^n} + \frac{1}{\beta^{n+1} + \beta^n - \alpha^n}
\]

(2.1.6)

which tends to 0 as \(x \to \infty\), \(x \to \infty\), if \(s \to -1\). Hence the integral in
(2.1.6) is convergent for \(s > -1\). Also it is easily verified that
\[
\int_{s}^{1} \frac{[x] - \frac{1}{x}}{\beta^{n+1}}\,dx = \frac{1}{s^{n+1} + \beta^n - \alpha^n} \quad (s < 0).
\]

Hence
\[
\theta(s) = \int_{s}^{1} \frac{[x] - \frac{1}{x}}{\beta^{n+1}}\,dx \quad (-1 < s < 0).
\]  

(2.1.7)

Now we have the Fourier series
\[
[x] - \frac{1}{x} = \sum_{n \neq 0} \frac{\sin 2\pi nx}{2\pi n},
\]

(2.1.8)

where \(n\) is not an integer. Substituting in (2.1.6), and integrating term by term, we obtain
\[
\theta(s) = \sum_{n \neq 0} \frac{1}{2\pi n} \int_{s}^{1} \sin 2\pi nx\,dx
\]

\[
- \sum_{n \neq 0} \frac{\sin 2\pi nx}{2\pi n} \left(\frac{x}{\beta^{n+1}}\right)
\]

\[
= \sum_{n \neq 0} \frac{2\pi y \sin \frac{y}{\beta^{n+1}} - \cos \frac{y}{\beta^{n+1}}}{2\pi n} (1 - \frac{1}{s}).
\]

(2.1.9)

i.e. (2.1.1). This is valid primarily for \(-1 < s < 0\). Here, however, the
right-hand side is analytic for all values of \(s\) such that \(s < 0\). It therefore
provides the analytic continuation of \(\theta(s)\) over the remainder of
the plane, and there are no singularities other than the pole already
encountered at \(s = 1\).

We have still to justify the term-by-term integration. Since the
series (2.1.7) is boundedly convergent, term-by-term integration over
any finite range is permissible. It is therefore sufficient to prove that
\[
\lim_{N \to \infty} \sum_{n \neq 0} \frac{1}{2\pi n} \int_{s}^{1} \sin 2\pi nx\,dx = 0 \quad (-1 < s < 0)
\]

Now
\[
\int_{s}^{1} \frac{\sin 2\pi nx}{\beta^{n+1}}\,dx = \frac{\cos 2\pi nx}{2\pi n} \left(\frac{x}{\beta^{n+1}}\right) - \frac{\cos 2\pi nx}{2\pi n} \left(\frac{x}{\beta^{n+1}}\right)
\]

\[
- \frac{1}{2\pi n} \int_{s}^{1} \frac{dx}{\beta^{n+1}} - \frac{1}{2\pi n} \int_{s}^{1} \frac{dx}{\beta^{n+1}}\]

and the desired result clearly follows.
The functional equation (2.1.1) may be written in a number of different ways. Changing $s$ into $1 - s$, it is

\[ \xi(1-s) = 2^{1-s} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \xi(s). \]  

(2.1.8)

It may also be written

\[ \xi(s) = x(s)\xi(1-s), \]  

(2.1.9)

where

\[ x(s) = 2^s \pi^{-s-\frac{1}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right). \]  

(2.1.10)

and

\[ x(1-s) = 1. \]  

(2.1.11)

Writing

\[ \xi(s) = \xi(1-s) x(s) \]  

(2.1.12)

it is at once verified from (2.1.8) and (2.1.9) that

\[ \xi(s) = \xi(1-s). \]  

(2.1.13)

Writing

\[ \Xi(s) = \xi(-s), \]  

(2.1.14)

we obtain

\[ \Xi(s) = \Xi(1-s). \]  

(2.1.15)

The functional equation is therefore equivalent to the statement that $\Xi(s)$ is an even function of $s$.

The approximation near $s = 1$ can be carried a stage farther; we have

\[ \xi(s) = \frac{1}{s-1} + \gamma + O(1/(s-1)), \]  

(2.1.16)

where $\gamma$ is Euler's constant. For by (2.1.4)

\[ \lim_{s \to 1} \frac{\xi(s) - \frac{1}{s-1}}{s-1} = \lim_{s \to 1} \int_{0}^{s-1} \frac{z^{s-2}}{z} \, dz = \frac{1}{\Gamma(s)} \int_{0}^{s-1} z^{s-2} \, dz = \Gamma(s). \]

2.2. A considerable number of variants of the above proof of the functional equation have been given. A similar argument was applied by Hardy,† not to $\xi(s)$ itself, but to the function

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\xi(s). \]  

(2.2.1)

† Hardy (9).
we obtain a proof not fundamentally distinct from the first proof given
here.† The formula can also be used to give a proof depending on
(1—2x−1)p(x).

Actually cases of Poisson’s formula enter into several of the following
proofs; (2.6.3) and (2.8.3) are both cases of Poisson’s formula.

2.4. Second method. The whole theory can be developed in another
way, which is one of Riemann’s methods. Here the fundamental
formula is

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx \quad (s > 1). \]

(2.4.1)

To prove this, we have for \( \sigma > 0 \)

\[ \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \, dy = \frac{\Gamma(s)}{\Gamma(s)}. \]

Hence

\[ \Gamma(s) \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-n \sigma} \, d\sigma = \int_0^\infty \frac{e^{-\sigma}}{\sigma^{s-1}} \, d\sigma \]

if the inversion of the order of summation and integration can be
justified; and this is so by absolute convergence if \( s > 1 \), since

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^\infty \frac{e^{-\sigma}}{\sigma^{s-1}} \, d\sigma = \Gamma(s) \zeta(s) \]

is convergent for \( s > 1 \).

Now consider the integral

\[ I(s) = \int_0^\infty \frac{e^{-\sigma}}{\sigma^{s-1}} \, d\sigma, \]

where the contour \( C \) starts at infinity on the positive real axis, encircles
the origin once in the positive direction, excluding the points \( \pm 2\pi i, \pm 4\pi i, \ldots, \) and returns to positive infinity. Here \( e^{\sigma} \) is defined as \( e^{\sigma-i\pi} \) when the logarithm is real at the beginning of the contour; thus \( \log|e| \)
varies from 0 to \( 2\pi \) round the contour.

We can take \( C \) to consist of the real axis from \( \infty \) to \( \rho \) (0 < \( \rho < 2\pi \)),
the circle \( |z| = \rho \), and the real axis from \( \rho \) to \( \infty \). On the circle,

\[ |e^{i\theta} - 1| = |e^{i\theta} - e^{i\theta}| = |\theta - 1| \leq \theta, \]

\[ |e^{i\theta} - 1| > A |\theta|, \]

† Marshall (9).

‡ Ingham, Prime Numbers, 40.

24 THE FUNCTIONAL EQUATION

Hence the integral round this circle tends to zero with \( \rho \) if \( s > 1 \). On
making \( \rho \to 0 \) we therefore obtain

\[ I(s) = -\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx + \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx \]

\[ = (2\pi i) \Gamma(s) \zeta(s), \]

\[ = 2\pi i \Gamma(s) \zeta(s). \]

Hence

\[ \zeta(s) = \frac{e^{-s\pi i} \Gamma(1-s)}{2\pi i} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx. \]

(2.4.2)

This formula has been proved for \( s > 1 \). The integral \( I(s) \), however,
is uniformly convergent in any finite region of the \( s \)-plane, and we define
an integral function of \( s \). Hence the formula provides the analytic
continuation of \( \zeta(s) \) over the whole \( s \)-plane. The only possible singularities
are the poles of \( \Gamma(1-s), \) viz. \( s = 1, 2, 3, \ldots \). We know already that \( \zeta(s) \) is regular at \( s = 2, 3, \ldots \) and in fact it follows as once from
Cauchy’s theorem that \( I(s) \) vanishes at these points. Hence the only
possible singularity is a simple pole at \( s = 1 \). Hence

\[ I(1) = \int_0^\infty \frac{d\sigma}{e^{\sigma} - 1} = -\frac{1}{\sigma - 1} + \ldots, \]

and

\[ \Gamma(1-s) = \frac{1}{s - 1} + \ldots. \]

Hence the residue at the pole is 1.

If \( s \) is any integer, the integrand in \( I(s) \) is one-valued, and \( I(s) \) can be
evaluated by the theorem of residues. Since

\[ \frac{1}{e^{\sigma} - 1} = \frac{1}{e^{\sigma} - 1} + B_1 \frac{e^{\sigma}}{1} + B_2 \frac{e^{2\sigma}}{2} + \ldots, \]

where \( B_1, B_2, \ldots \) are Bernoulli’s numbers, we find the following values
of \( \zeta(s) \):

\[ \zeta(0) = -\frac{1}{2}, \quad \zeta(2m) = -\frac{1}{2} \pi^2 B_{2m}, \quad \zeta(2m + 1) = -B_{2m}. \]

To deduce the functional equation from (2.4.2), take the integral
along the contour \( C_0 \) consisting of the positive real axis from infinity
to \((2n+1)x\), then round the square with corners \((2n+1)x(\pm 1-i, i)\), and
then back to infinity along the positive real axis. Between the contours
\[ C \text{ and } C_n \text{ the integrand has poles at the points } \pm 2\pi m, \pm 2i\pi n. \text{ The residues at } 2\pi im \text{ and } -2\pi im \text{ are together} \]
\[ 2\pi ime^{(t^{c*})^{-1}+2\pi ime^{(t^{c*})^{-1}}-2\pi ime^{(t^{c*})^{-1}}\cos \frac{\pi}{2}(e-1)} - 2\pi ime^{(-t^{c*})^{-1}} - 2\pi ime^{(-t^{c*})^{-1}}\cos \frac{\pi}{2}(e-1) \]

Hence by the theorem of residues
\[ I(s) = \int_{C} \frac{e^{zt^{c*}}} {\Gamma(t)} \sin \frac{\pi s}{2} \frac{\sin \pi \frac{s}{2}} {\sin \pi \frac{s}{2}} \sum_{n=1}^{\infty} (2\pi m)^{-1}. \]

Now let \( \sigma < 0 \) and make \( n \to \infty \). The function \( 1/(c^{t^{c*}}-1) \) is bounded on the contours \( C_n \) and \( t^{c*} = O(|z|^{-1}) \). Hence the integral round \( C_n \) tends to zero, and we obtain
\[ I(s) = 4\pi e^{\pi i \sigma} \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} (2\pi m)^{-1} \]
\[ - 4\pi e^{\pi i \sigma} \sin \pi \frac{s}{2} \mathcal{Z}(2\pi m)^{-1}. \]

The functional equation now follows again.

Two more consequences of the functional equation may be noted here. The formula
\[ \frac{\Gamma(2m)} {2\pi m} = 2^{2m-1} \pi \sin \frac{2\pi m}{2m} \]
follows from the functional equation (2.1.1), with \( s = 1-2m \), and the value obtained above for \( \frac{\Gamma(1-2m)} {2\pi m} \). Also
\[ \mathcal{Z}(0) = -1 \log 2\pi. \quad (2.4.4) \]

For the functional equation gives
\[ \frac{\mathcal{Z}(1+s)} {\mathcal{Z}(1-s)} = -\log 2\pi - \frac{\sin \pi s}{\sin \pi \frac{s}{2}} \mathcal{Z}(s) + \mathcal{Z}(1-s). \quad (2.4.5) \]

In the neighbourhood of \( s = 1 \)
\[ \frac{\mathcal{Z}(1+s)} {\mathcal{Z}(1-s)} = -\log 2\pi - \frac{\sin \pi s}{\sin \pi \frac{s}{2}} \mathcal{Z}(s) + \mathcal{Z}(1-s), \quad (2.4.6) \]
and
\[ \mathcal{Z}(s) \mathcal{Z}(1-s) = \frac{1}{2(1-s)} + \frac{1}{2s} - y + \cdots, \]
where \( y \) is a constant. Hence, making \( s \to 1 \), we obtain
\[ \frac{\mathcal{Z}(0)} {\mathcal{Z}(0)} = -\log 2\pi, \]
and (2.4.5) follows.

2.5. Validity of (2.2.1) for all \( s \). The original series (1.1.1) is naturally valid for \( \sigma > 1 \) only, on account of the pole at \( s = 1 \). The series (2.2.1) is convergent, and represents \( (1-2^{1-s})\mathcal{Z}(s) \), for \( \sigma > 0 \). This series ceases to converge on \( \sigma = 0 \), but there is nothing in the nature of the function represented to account for this. In fact if we use summability instead of ordinary convergence the equation still holds to the left of \( \sigma = 0 \).

Theorem 25. The series \( \sum_{n=1}^{\infty} (1-n^{1-s}) \mathcal{Z}(s) \) is convergent for all values of \( s \).

Let \( 0 < \sigma < 1 \). Then
\[ \sum_{n=1}^{\infty} (1-n^{1-s}) \mathcal{Z}(s) = \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} e^{-n^{1-s}} \mathcal{Z}(s) = \Gamma(s) \int_{0}^{\infty} \frac{e^{-x^{1-s}} \mathcal{Z}(s)} {1+x} dx. \]

This is justified by absolute convergence for \( \sigma > 1 \), and the result by analytic continuation for \( \sigma > 0 \).

We can now replace this by a loop-integral in the same way as (2.4.2) was obtained from (2.4.1). We obtain
\[ \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} = \frac{1}{\Gamma(s)} \int_{1}^{\infty} \frac{x^{1-s} \mathcal{Z}(s)} {1+x^{1-s}} dx, \]
when \( C \) encircles the origin as before, but excludes all zeros of \( 1+x^{1-s} \), i.e. the points \( w = \log z/(2\pi m+1) \).

It is clear that, as \( s \to 1 \), the right-hand side tends to a limit, uniformly in any finite region of the \( \sigma \)-plane containing positive integers, and, by the theory of analytic continuation, the limit must be \( (1-2^{1-s})\mathcal{Z}(s) \). This proves the theorem except if \( s \) is a positive integer, when the proof is elementary.

Similar results hold for other methods of summation.

2.6. Third method. This is also one of Riemann's original proofs.

We observe that if \( s > 0 \)
\[ \int_{0}^{\infty} x^{s-1} \mathcal{Z}(s) ds = \frac{\Gamma(s)} {s^{1-s}}. \]

Hence if \( s > 1 \)
\[ \frac{\Gamma(s)} {s^{1-s}} = \int_{0}^{\infty} x^{s-1} e^{-x} dx = \int_{0}^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-n x} dx, \]
the inversion being justified by absolute convergence, as in § 2.4.
Writing
\[ \psi(x) = \sum_{n=1}^\infty \frac{\sin nx}{n^s} \]
we therefore have
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-sx} \psi(x) \, dx \quad (s > 1). \]  
(2.6.1)

Now it is known that, for \( x > 0, \)
\[ \sum_{n=1}^\infty e^{-nx} = \frac{1}{x} \sum_{n=1}^\infty e^{-nx}, \]
\[ 2\Phi(x) - 1 = \frac{1}{\sqrt{n}} \left( \psi \left( \frac{n}{\sqrt{n}} \right) \right) \]
(2.6.2)

Hence (2.6.2) gives
\[ e^{i\Gamma(s)} = \int_0^\infty e^{i\Phi(x)} \, dx + \int_0^\infty e^{i\Phi(x)} \, dx \]
\[ = \int_0^\infty e^{i\Phi(x)} \, dx + \frac{1}{2} \int_0^\infty e^{i\Phi(x)} \, dx + \frac{1}{2} \int_0^\infty e^{i\Phi(x)} \, dx \]
\[ = \frac{1}{2} \int_0^\infty e^{i\Phi(x)} \, dx + \frac{1}{2} \int_0^\infty e^{i\Phi(x)} \, dx \]
\[ = \frac{1}{2} \int_0^\infty e^{i\Phi(x)} \, dx. \]
(2.6.3)

The last integral is convergent for all values of \( s \), and so the formula holds, by analytic continuation, for all values of \( s \). Now the right-hand side is unchanged if \( s \) is replaced by \( 1 - s \). Hence
\[ e^{i\Gamma(s)} = e^{i\Gamma(1-s)} \]
(2.6.4)

which is a form of the functional equation.

2.7 Fourth method: proof by self-reciprocal functions. Still another proof of the functional equation is as follows. For \( s > 1 \), (2.4.1) may be written
\[ \zeta(s) \Gamma(s) = \int_1^\infty \left( \frac{1}{t-1} - \frac{1}{t-\frac{s}{2}} \right) dt \]
and this holds by analytic continuation for \( s > 0 \). Also for \( 0 < s < 1 \)
\[ \frac{1}{s-1} = - \int_\frac{s}{2}^{\infty} \frac{\sin t}{t} \, dt. \]
(2.7.1)

Hence
\[ \zeta(s) \Gamma(s) = \int_1^\infty \left( \frac{1}{t-1} - \frac{1}{t-\frac{s}{2}} \right) dt + \int_\frac{s}{2}^{\infty} \frac{\sin t}{t} \, dt \quad (0 < s < 1). \]
(2.7.2)

Now it is known that the function
\[ f(x) = \frac{1}{x^{\alpha+1}} \left( \frac{1}{x} - 1 \right) \]
is self-reciprocal for sine transforms, i.e.
\[ f(x) = \int_0^\infty \frac{f(y)}{y^{\alpha+1}} \sin \theta y \, dy. \]
(2.7.3)

Hence, putting \( x = \frac{\pi}{2} \) in (2.7.1),
\[ \zeta(s) \Gamma(s) = (2\pi)^{1-s} \int_0^\infty f(y) y^{s-1} \, dy \]
\[ - \frac{1}{\alpha} \int_0^\infty \left( \frac{1}{y} \sin \theta y \right) y^{s-1} \, dy \]
(2.7.4)

If we can invert the order of integration, this is
\[ 2\pi \int_0^\infty \left( \frac{1}{y} \sin \theta y \right) y^{s-1} \, dy \]
\[ = 2\pi \int_0^\infty \left( \frac{1}{y} \sin \theta y \right) y^{s-1} \, dy \]
\[ = 2\pi \int_0^\infty \left( \frac{1}{y} \sin \theta y \right) y^{s-1} \, dy \]
and the functional equation again follows.

To justify the inversion, we observe that the integral
\[ \int_0^\infty f(y) \sin \theta y \, dy \]
converges uniformly over \( 0 < \theta < \pi \), \( \Delta \). Hence the inversion of this part is valid, and it is sufficient to prove that
\[ \lim_{\theta \to \infty} \int_0^\pi f(y) \sin \theta y \, dy = 0. \]

Now
\[ \int_0^\infty 2\pi \sin \theta y \, dy = \int_0^\infty \cos \theta y \, dy = \int_0^\pi \cos \theta y \, dy \]
and also
\[ = y^{\frac{1}{2}} \int_0^\infty \sin \theta y \, dy = O(y^{-\alpha+1}). \]
Since \( f(y) = O(1) \) as \( y \to 0 \), and \( = O(y^{-1}) \) as \( y \to \infty \), we obtain

\[
\int y f(y) \, dy = -\int y \, dx + \int O(y^{2}) \, dy \to 0.
\]

A similar method shows that the integral involving \( \Delta \) also tends to 0.

2.8. Fifth method. The process by which (2.7.1) was obtained from (2.4.1) can be extended indefinitely. For the next stage, (2.7.1) gives

\[
\frac{\Gamma(s)}{\Gamma(s+1)} = \frac{\int \left( \frac{1}{x^s-1} - \frac{1}{x^{s-1}} \right) dx}{\int \left( \frac{1}{x^s} - \frac{1}{x^{s-1}} \right) x^{s-1} \, dx},
\]

and this holds by analytic continuation for \( s > 1 \). But

\[
\int \frac{1}{x^s} \, dx = -\frac{1}{2} \left( -1 < s < 0 \right).
\]

Hence

\[
\frac{\Gamma(s)}{\Gamma(s+1)} = \frac{\int \frac{1}{x^s-1} - \frac{1}{x^{s-1}} dx}{\int \frac{1}{x^s} - \frac{1}{x^{s-1}} x^{s-1} \, dx},
\]

\[
\int \frac{1}{x^s-1} dx = \frac{1}{2} \left( -1 < s < 0 \right).
\]

Now

\[
\frac{\Gamma(s)}{\Gamma(s+1)} = \frac{\int \frac{1}{x^s} + \frac{1}{x^{s+1}} dx}{\int \frac{1}{x^s} - \frac{1}{x^{s+1}} x^{s+1} \, dx},
\]

\[
\int \frac{1}{x^s} dx = \frac{1}{2} \left( -1 < s < 0 \right).
\]

Hence

\[
\frac{\Gamma(s)\Gamma(1-s)}{\pi} = \frac{\int \frac{1}{x^s} + \frac{1}{x^{s+1}} dx}{\int \frac{1}{x^s} - \frac{1}{x^{s+1}} x^{s+1} \, dx}.
\]

the functional equation. The inversion is justified by absolute convergence if \(-1 < s < 0\).

2.9. Sixth method. The formula is easily proved by the calculations of residues if \( \varepsilon > 1 \), and the integral is \( O(\varepsilon^{-1}) \), so that the integral is convergent, and the formula holds by analytic continuation, if \( \varepsilon > 0 \).

\[\Phi(s) = \Phi(s+1) \quad \text{at} \quad s = 0.\]
Hence, using the formula for the derivative of the exponential function,

\[ \Phi''(z) = \frac{d}{dz} \left( e^{z} \right) = e^{z} \]

we have

\[ \Phi(z) = \int e^{z} \, dz = e^{z} + C. \]

But since \( \Phi(0) = 1 \), we conclude that \( C = 1 \), and

\[ \Phi(z) = e^{z}. \]

Next, let \( L \) be the line

\[ \{ z = a + b \, e^{i \theta} : 0 < \theta < \pi \} \]

and let \( \gamma \) be the contour in the partial line through the origin, so that the last integral is

\[ \int_{\gamma} \Phi(z) \, dz. \]

This formula holds by the theory of analytic continuation for all values of \( a \) and \( b \) of the two terms on the right are continuous.

Hence

\[ \int_{\gamma} \Phi(z) \, dz = \int_{\gamma} e^{z} \, dz = e^{z}. \]

This formula holds by the theory of analytic continuation for all values of \( a \) and \( b \) of the two terms on the right are continuous.

Hence

\[ \int_{\gamma} \Phi(z) \, dz = \int_{\gamma} e^{z} \, dz = e^{z}. \]

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Hence

\[ \int_{\gamma} \Phi(z) \, dz = \int_{\gamma} e^{z} \, dz = e^{z}. \]

This formula holds by the theory of analytic continuation for all values of \( a \) and \( b \) of the two terms on the right are continuous.

Hence

\[ \int_{\gamma} \Phi(z) \, dz = \int_{\gamma} e^{z} \, dz = e^{z}. \]
2.11. A general formula involving \( \xi(x) \). It was observed by Mittag-Leffler that several of the formulas for \( \xi(x) \) which we have obtained are particular cases of a formula containing an arbitrary function.

We have formally

\[
\int_0^\infty \sum_{n=1}^\infty f(nx) \, dx = \sum_{n=1}^\infty \int_0^\infty f(nx) \, dx = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty g(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx,
\]

valid for \( 0 < \alpha < 1 \) if \( f(x) \) satisfies the above conditions.

If \( f(x) = e^{-x} \) we obtain (2.7.1); if \( f(x) = e^{-x^2} \) we obtain a formulas equivalent to those of § 2.6; if \( f(x) = 1/(1+x^2) \) we obtain a formula which is also obtained by combining (2.4.1) with the functional equation. If \( f(x) = x^{-\alpha} \sin \pi x \) we obtain a formulas equivalent to (2.6.6), though this \( f(x) \) does not satisfy our general conditions.

If \( f(x) = 1/(1+x^2) \), we have

\[
\sum_{n=1}^\infty \frac{1}{x} \int_0^\infty f(x) \, dx = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx.
\]

Hence

\[
\frac{1}{\sin \pi x} \int_0^\infty f(x) \, dx = \int_0^\infty \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx.
\]

and on integrating by parts we obtain (2.9.9).

2.12. Zeros; factorization formulæ.

**Theorem 2.12.** \( \xi(x) \) and \( \zeta(s) \) are integral functions of order 1.

It follows from (2.11.1) and what we have proved about \( \xi(x) \) that \( \xi(x) \) is regular for \( x > 0 \), \( (x-1)\xi(x) \) being regular at \( x = 1 \). Since \( \xi(x) = \xi(1-x) \), \( \xi(x) \) is also regular for \( x < 1 \). Hence \( \xi(x) \) is an integral function.

Also

\[
\frac{1}{\sin \pi x} \int_0^\infty f(x) \, dx = \int_0^\infty \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx.
\]

and (2.1.4) gives for \( x > 1 \), \( |x-1| < A \),

\[
\xi(x) = \frac{1}{2} \left( \frac{2}{x} \right) \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx.
\]

Hence, as \( x \to \infty \), 

\[
\xi(x) = \frac{1}{2 \pi} \Gamma(x) \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx \frac{1}{x} \int_0^\infty f(x) \, dx.
\]

for \( x > \frac{1}{2} \), \( |x| > A \). By (2.1.12) this holds for \( x < \frac{1}{2} \) also. Hence \( \xi(x) \) is of order 1 at most. The order is exactly 1 since as \( x \to \infty \) by real values \( \log \xi(x) \sim 2 \pi, \log \xi(x) \sim 4 \pi \) log x.

\( \dagger \) Mittag-Leffler.
Hence also \( \Xi(z) = O(e^{\rho |\Re z|}) \) \((|z| > A)\),
and \( \Xi(z) \) is of order 1. But \( \Xi(z) \) is an even function. Hence \( \Xi(z) \) is also an integral function, and is of order \( \frac{1}{2} \). It has therefore an infinity of zeros, whose exponent of convergence is \( \frac{1}{2} \). Hence \( \Xi(z) \) has an infinity of zeros, whose exponent of convergence is 1. The same is therefore true of \( \zeta(z) \). Let \( \rho_1, \rho_2, \ldots \) be the zeros of \( \zeta(z) \).

We have already seen that \( \zeta(z) \) has no zeros for \( \sigma > 1 \). It then follows from the functional equation (2.1.1) that \( \zeta(z) \) has no zero for \( \sigma < 0 \) except for simple zeros at \( z = -1 - 2\pi i \frac{1}{2}, \ldots \); for, in (2.1.1), \( \Omega(s) \) has no zeros for \( \sigma < 0 \), since \( \Psi(\sigma) \) has simple zeros at \( z = -2, 4, \ldots \) only, and \( \Gamma(1-s) \) has no zeros.

The zeros of \( \zeta(z) \) at \( -2, -4, \ldots \), are known as the 'trivial zeros'. They do not correspond to zeros of \( \zeta(z) \), since in (2.1.12) they are cancelled by poles of \( \Gamma(\sigma) \). It therefore follows from (2.1.12) that \( \zeta(z) \) has no zeros for \( \sigma = 1 \) or for \( \sigma < 1 \). Its zeros \( \rho_1, \rho_2, \ldots \), therefore all lie in the strip \( 0 < \sigma < 1 \); and they are also zeros of \( \zeta(s) \), since \( \sigma(1-s) \Gamma(s) \) has no zeros in the strip except at \( s = 1 \), which is cancelled by the pole of \( \Gamma(1-s) \).

We have thus proved that \( \zeta(s) \) has an infinity of zeros \( \rho_1, \rho_2, \ldots \) in the strip \( 0 < \sigma < 1 \).

\[
(1 - 2\pi i \sigma) \zeta(s) = 1 - \frac{1}{2^s} + \frac{\pi \tan \left( \frac{\pi s}{2} \right)}{2^{\frac{1}{2} s} \sqrt{\pi}} \Gamma(s) \zeta(s) \geq 0 \quad (0 < \sigma < 1)
\]  
(2.12.4)

and \( \zeta(0) \neq 0 \), \( \zeta(s) \) has no zeros on the real axis between 0 and 1. The zeros \( \rho_1, \rho_2, \ldots \) are therefore all complex.

The remainder of the theory is largely concerned with questions about the position of these zeros. At this point we shall merely observe that they are in conjugate pairs, since \( \zeta(s) \) is real on the real axis; and that, if \( \rho \) is a zero, so is \( 1 - \rho \), by the functional equation, and hence so is \( 1 - \rho \). If \( \rho = \beta + i \gamma \), then \( 1 - \rho = 1 - \beta + i(-\gamma) \). Hence the zero either lies on \( \sigma = \frac{1}{2} \), or occurs in pairs symmetrical about this line.

Since \( \xi(s) \) is an integral function of order 1, and \( \xi(0) = \frac{1}{12} \), Hadamard's factorization theorem gives, for all values of \( s \),

\[
\xi(s) = \exp \left( 1 + \sum_{\rho} \frac{s}{s - \rho} \right)
\]  
(2.12.5)

where \( \beta \) is a constant. Hence

\[
\xi(s) = \exp \left( \beta s \right) 2^{s(1-\frac{1}{2})} \prod_{\rho} \left( 1 - \frac{s}{s - \rho} \right)
\]  
(2.12.6)

2.12. THE FUNCTIONAL EQUATION

where \( b = b_0 + \frac{1}{2} \log \pi \). Hence also

\[
\zeta(s) \zeta(1 - s) = \frac{\Gamma(s) \Gamma(1 - s)}{\pi^{s - \frac{1}{2}}} \frac{1}{\sin \pi s}
\]

(2.12.7)

Making \( s \to 0 \), this gives

\[
\zeta' \zeta(1 - s) \zeta(1 - s) \zeta(1 - s) \zeta(1 - s) = \frac{1}{\pi^{s - \frac{1}{2}}} \frac{1}{\sin \pi s}
\]

(2.12.8)

2.13. In this section we shall show that the only function which satisfies the functional equation (2.1.1), and has the same general characteristics as \( \xi(s) \), is \( \zeta(s) \) itself.

Let \( G(s) \) be an integral function of finite order, \( P(s) \) a polynomial, and \( f(s) = G(s)P(s) \), and let

\[
f(s) = \sum_{n=0}^{\infty} a_n s^n
\]

(2.13.1)

be absolutely convergent for \( \sigma > 1 \). Let

\[
f(s) \Gamma(1-s) = g(1-s) \Gamma(1-s) e^{-\pi i \sigma}
\]

(2.13.2)

where

\[
g(1-s) = \sum_{n=0}^{\infty} \frac{a_n}{s^n}
\]

the series being absolutely convergent for \( \sigma < -a < 0 \). Then \( f(s) = C \zeta(s) \), where \( C \) is a constant.

We have, for \( x > 0 \),

\[
\phi(s) = \frac{1}{2\pi i} \int_{C - R}^{C + R} f(s) \Gamma(1-s) e^{-\pi i \sigma} ds
\]

\[
= \sum_{n=0}^{\infty} a_n \Gamma(s) \Gamma(1-s) e^{-\pi i \sigma} ds
\]

\[
= 2 \pi i \sum_{n=0}^{\infty} a_n \Gamma(s) \Gamma(1-s) e^{-\pi i \sigma} ds
\]

(2.13.3)

Also, by (2.13.2),

\[
\phi(s) = \frac{1}{2\pi i} \int_{C - R}^{C + R} g(s) \Gamma(1-s) e^{-\pi i \sigma} ds
\]

We move the line of integration from \( s = 2 \) to \( s = 1 - a \). We observe *Hamburger* (1)-(4), *Siegel* (1).
that \( f(s) \) is bounded on \( \sigma = 2 \), and \( g(1-s) \) is bounded on \( \sigma = -1-s \); since
\[
\Gamma(s+1) = \Gamma(s) + \frac{\Gamma(s)}{s} = O((1-s)^{-1}),
\]
it follows that \( g(1-s) = O((1-s)^{-1}) \) on \( \sigma = 2 \). We can therefore, by the
Perron-Fleischer-Lindelöf principle, apply Cauchy's theorem, and obtain
\[
\phi(s) = \int_{C} g(1-s) (t-t)^{-1} e^{-st} dt = \sum_{n=0}^{\infty} R_n,
\]
where \( R_n, R_{n+1}, \ldots \) are the residues at the poles, say \( a_1, \ldots, a_n \). Thus
\[
\sum_{n=0}^{\infty} R_n = \sum_{n=0}^{\infty} Q_n \left( \log x \right) = Q(x),
\]
where the \( Q_n \) are polynomials in \( \log x \). Hence
\[
\phi(s) = \frac{1}{\zeta(1-s)} \sum_{n=0}^{\infty} b_n e^{2\pi i n \sigma} \frac{1}{s-n} + Q(x).
\]
Multiply by \( e^{-\pi x} \) (\( s > 0 \)), and integrate over \((0, \infty)\). We obtain
\[
\sum_{n=0}^{\infty} \frac{b_n}{s-n} e^{2\pi i n \sigma} \frac{1}{s-n} = \sum_{n=0}^{\infty} b_n e^{-\pi x} + Q(x),
\]
and the last term is a sum of terms of the form
\[
\int \pi x \log^2(1+\pi x) dx,
\]
where the \( b_n \) are integers and \( R_n > -1 \); i.e., it is a sum of terms of the form \( e^{\pi x} \).

Hence
\[
\sum_{n=0}^{\infty} \frac{a_n}{s-n} e^{-\pi x} = \sum_{n=0}^{\infty} a_n e^{-\pi x},
\]
where \( H(t) \) is a sum of terms of the form \( e^{\pi x} \).

Now the series on the left is a meromorphic function, with poles at \( \pm n \). But the function on the right is periodic, with period \( n \). Hence (by analytic continuation) so is the function on the left. Hence the poles at \( k \) and \((k+1)\) are equal, i.e., \( a_k = a_{k+1} \) (\( k = 1, 2, \ldots \)). Hence \( a_k = a_1 \) for all \( k \), and the result follows.
and from (2.6.2) the pair
\[ f(x) = \phi(x), \quad \theta(x) = e^{-\Gamma(s)x^2} \text{ for } \sigma > \frac{1}{2}. \] (2.15.4)

The inverse formulas are thus
\[ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)x^{-s} \, ds = \frac{1}{\sigma - 1} \quad (\sigma > 1) \] (2.15.5)
and
\[ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\Gamma(s)2z^2} \, ds = \phi(z) \quad (\sigma > \frac{1}{2}). \] (2.15.6)

Each of these can easily be proved directly by inserting the series for \( f(x) \) and integrating term-by-term, using (2.15.2).

As another example, (2.6.2), with \( s \) replaced by \( 1 - s \), gives the Mellin pair
\[ f(x) = \frac{\Gamma(1+x)}{\Gamma(1+x)} \log x, \quad \theta(s) = \frac{-\pi(1-s)}{\sin \pi s} \quad (0 < s < 1). \] (2.15.7)

The inverse formula is thus
\[ \frac{\Gamma(1+x)}{\Gamma(1+x)} \log x = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log(1-s)}{\sin \pi s} \, ds. \] (2.15.8)

Integrating with respect to \( x \), and replacing \( x \) by \( 1 - x \), we obtain
\[ \log \Gamma(1-x) - x \log x - x = -\frac{1}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\log(s)x}{\sin \pi s} \, ds \quad (0 < s < 1). \] (2.15.9)

This formula is used by Whittaker and Watson to obtain the asymptotic expansion of \( \log \Gamma(1-x) \).

Next, let \( f(x) \) and \( \theta(x) \) be related by (2.15.1), and let \( g(x) \) and \( \theta(x) \) be similarly related. Then we have, subject to appropriate conditions,
\[ \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \theta(s) \theta(s-x) \, ds = \frac{1}{2\pi} \int_{0}^{\infty} \theta(\xi) g(\xi) e^{-\xi x} \, d\xi. \] (2.15.10)

Take for example \( \theta(s) = \Gamma(s) \frac{1}{\Gamma(1+s)} \), so that
\[ f(x) = g(x) = \frac{1}{(s-1)}. \]

Then, if \( R(w) > 2 \), the right-hand side is
\[ \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{-\Gamma(s)z^2}}{(e^{R(w)} - 1)^2} \, ds = \frac{1}{2\pi} \int_{0}^{\infty} \left( \frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \ldots \right) \theta(\xi) g(\xi) \, d\xi = \Gamma(w) \left[ (1 + \xi) - (\xi - 1) \right]. \]

Thus if \( 1 < \epsilon < R(w) - 1 \)
\[ \frac{1}{2\pi} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(w-e^{-s})}{\Gamma(1+w-e^{-s})} \, ds = \Gamma(w) \left[ (1 + \xi) - (\xi - 1) \right]. \] (2.15.11)

Similarly, taking \( \theta(s) = \Gamma(s) \frac{\Gamma(2s)}{\Gamma(1+s)} \), so that
\[ f(x) = g(x) = \phi(x) = \frac{\pi}{2\pi x}, \]
the right-hand side of (2.16.5) is, if \( R(w) > 1 \),
\[ \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(2s)}{\Gamma(1+s)} \, ds = \Gamma(w) \sum_{n=1}^{\infty} \frac{1}{\Gamma(w+n)} \sum_{m=1}^{\infty} \frac{1}{\Gamma(w+m+n)}. \]

This may also be written
\[ \Gamma(w) \left[ \frac{\pi}{2\pi x} \right] = \frac{1}{2\pi} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(2s)}{\Gamma(1+s)} \, ds = \Gamma(w) \Gamma(2w) \Gamma(w-1). \]

Hence\footnote{Hardy (4). A generalization is given by Taylor (1).} if \( 1 < \epsilon < R(w) - 1 \)
\[ \frac{1}{2\pi} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(2s)}{\Gamma(1+s)} \, ds = \Gamma(w) \Gamma(2w) \Gamma(w-1). \] (2.16.12)

2.16. Some integrals involving \( \Xi(t) \). There are some complex integrals of the form
\[ \frac{1}{2\pi} \int_{\epsilon - i\infty}^{\epsilon + i\infty} f(t) e^{\xi t} \, dt \]
can be evaluated. Let \( f(t) = \frac{1}{2\pi} |\phi(t)|^2 = \phi(t) \overline{\phi(-t)} \), where \( \phi \) is analytic. Writing \( y = e^{\xi t} \),
\[ \Xi(t) = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(-t, y) e^{-\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ + \int_{-\infty}^{\infty} \phi(-t, y) e^{-\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ + \int_{-\infty}^{\infty} \phi(-t, y) e^{-\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ + \int_{-\infty}^{\infty} \phi(-t, y) e^{-\xi t} \, dt \]
\[ = \int_{-\infty}^{\infty} \phi(t, y) e^{\xi t} \, dt \]
\[ + \int_{-\infty}^{\infty} \phi(-t, y) e^{-\xi t} \, dt \]
Taking \( \psi(s) = 1 \), this is equal to

\[
\frac{1}{i\nu y} \sum_{n = 1}^{\infty} \left( \Gamma(1 - s) \right) \frac{y}{|n/v|} \gamma^s \ dx
\]

\[
= \frac{1}{i\nu y} \sum_{n = 1}^{\infty} \left[ 2 \Gamma(1 - s) \frac{y}{|n/v|} \gamma^s \ dx - 2 \Gamma(1 - s) \frac{y}{|n/v|} \gamma^s \ dx \right]
\]

\[
= \frac{2\pi}{i\nu y} \sum_{n = 1}^{\infty} \left( \frac{y}{|n/v|} \right) \gamma^s \ e^{\pi i n/v} \ dx
\]

Hence

\[
\int \frac{\mathcal{L}(t) \cos dt \ dz}{\beta^2 - \gamma^2} = 2\pi y \sum_{n = 1}^{\infty} \left( 2\pi y^2 - 12\pi y - y^2 - 12\pi y \right) \exp(-n^2 y^2). \tag{2.10.1}
\]

Again, putting \( \psi(s) = 1/(s+1) \), we have

\[
\Phi(z) = -\frac{1}{2i\nu y} \int_{1-i\infty}^{1+i\infty} \Gamma(1 + s) \xi(s) \gamma^s \ dx
\]

\[
= -\frac{\pi}{4i\nu y} \int_{1-i\infty}^{1+i\infty} \Gamma(1 + s) \xi(s) \gamma^s \ dx
\]

\[
= -\frac{\pi}{4i\nu y} \frac{1}{(1+y)^2} \tag{2.16.2}
\]

in the notation of § 2.6. Hence

\[
\int \frac{\mathcal{L}(t) \cos dt \ dz}{\beta^2 - \gamma^2} = \int \mathcal{L}(t) \cos dt \ (e^{\beta t} - 2\pi y \mathcal{L}(e^{-\beta t})). \tag{2.16.2}
\]

The case \( \psi(0) = \Gamma(1 - s) \) was also investigated by Ramanujan, the result being expressed in terms of another integral.

2.17. The function \( \xi(s, a) \). A function which is in some sense a generalization of \( \zeta(s) \) is the Hurwitz zeta-function, defined by

\[
\xi(s, a) = \sum_{n = 1}^{\infty} \frac{1}{(n + a)^s} \ (0 < a < 1, \ s > 1).
\]

This reduces to \( \zeta(s) \) when \( a = 1 \), and to \( (s - 1)\zeta(s) \) when \( a = 1 \). We shall obtain here its analytic continuation and functional equation, which are required later. This function, however, has no Euler product unless \( a = \frac{1}{2} \) or \( a = 1 \), and so does not share the most characteristic properties of \( \zeta(s) \).

2.18. Selberg [11] has given a very general method for obtaining the analytic continuation and functional equation of certain types of zeta-function which arise as the 'constant terms' of Eisenstein series. We sketch a form of the argument in the classical case. Let \( \mathcal{H} = \{ -y + iy \} \) be the upper half plane and define

\[
B(z, s) = \sum_{\gamma \in \mathcal{H}} \frac{y^s}{(\gamma + 1)^s} \ (z \in \mathcal{H}, \ s > 1)
\]

and

\[
B(z, s) = \frac{\zeta(s)}{\zeta(s - 1)} \prod_{\gamma \in \mathcal{H}} \frac{1}{1 - y^s} \ (z \in \mathcal{H}, \ s > 1),
\]

these series being absolutely and uniformly convergent in any compact subset of the region \( \Re(s) > 1 \). Here \( B(z, s) \) is an Eisenstein series, while \( \zeta(s) \) is, apart from the factor \( y^s \), the Epstein zeta-function for the lattice generated by 1 and \( z \). We shall find it convenient to work with \( B(z, s) \) in preference to \( \zeta(s) \).
We begin with two basic observations. Firstly one trivially has
\[ B(z+1, s) - B(1, s) = -B(z, s) \]  
(Thus, in fact, \( B(z, s) \) is invariant under the full modular group.) Secondly, if \( \Delta \) is the Laplace–Beltrami operator
\[ \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]
then
\[ \Delta \left( \frac{y^2}{|cz+d|^4} \right) = -y^2 \frac{y^2}{|cz+d|^4}, \]
whence
\[ \Delta B(z, s) = s(1-s)B(z, s) \quad (s > 1). \]  
(2.18.3)
We proceed to obtain the Fourier expansion of \( B(z, s) \) with respect to \( z \).
We have
\[ B(z, s) = \sum_{c \in \langle \mathbb{Z} \times \mathbb{Z} \rangle} a(c, s) e^{2\pi i cz}. \]
where
\[ a(c, s) = \sum_{c \in \mathbb{Z}^2} \frac{e^{-2\pi i cz}}{e^{2\pi i cz + i \overline{cz} + i cz}} \]
with \( \delta_c = 1 \) or 0 according as \( n = 0 \) or not. The \( d \) summation above is
\[ \sum_{c \in \mathbb{Z}^2} \sum_{d \in \mathbb{Z}^2} \frac{e^{-2\pi i cd}}{e^{2\pi i cd + i \overline{cd} + i cd}} = \sum_{c \in \mathbb{Z}^2} \sum_{d \in \mathbb{Z}^2} \frac{e^{-2\pi i cd}}{e^{2\pi i cd + i \overline{cd} + i cd}} \]
and the sum over \( k \) is \( c \) or 0 according as \( c \) is or not. Moreover
\[ \int_{\infty}^0 \frac{dz}{(z^2 + 1)^s} = \pi \Gamma(s-1) \Gamma(s), \]
and
\[ \int_{\infty}^0 \frac{dz}{(z^2 + 1)^s} = \pi \Gamma(s-1) \Gamma(s) \]
In the usual notation of Bessel functions,
We now have
\[ B(z, s) = \phi(\delta) \psi(s) \Gamma(s-1) + B_1(z, s) \]
where
\[ \phi(\delta) = \zeta(2s), \quad \psi(s) = 2\Gamma(s-1) \Gamma(s) \]
and
\[ B_1(z, s) = \sum_{n=1} \frac{n^{s-1} \pi \Gamma(s)}{\Gamma(s) \Gamma(s-1)} \cos(2\pi n x). \]
(2.18.5)
We observe at this point that
\[ K_{\delta}(z) \equiv \Gamma(s-1) \Gamma(s) \]
for fixed \( u \), whence the series (2.18.5) is convergent for all \( s \), and defines an entire function. Moreover we have
\[ B_1(z, s) \equiv \Gamma(s) \Gamma(s-1) \]
for fixed \( s \). Similarly one finds
\[ \partial B(z, s) / \partial y \equiv e^{-z} \quad (y \to \infty) \]
(2.18.6)
We proceed to derive the 'Masse-Selberg' formula. Let \( D = \{ x \in \mathbb{Z} : x > 1, \{(x, d) \in \mathbb{Z} \} \} \) be the standard fundamental region for the modular group, and let \( D_1 = \{ x \in D : \langle x \rangle \} \). Let \( B(s) \), \( R(u) \) be \( > 1 \) and write, for convenience, \( F = B(z, s), \mathcal{G} = B(z, w) \). Then, according to (2.18.3), we have
\[ \int_{D_1} (s(1-s) - x(1-x)) \int_{\mathcal{G}} (GF - \mathcal{G}F) \frac{dx dy}{y^2} = \int_{\mathcal{G}} (PVGF - GVF) \frac{dx dy}{y^2}, \]
and
\[ \int_{\mathcal{G}} (FVG - GFV) \frac{dx dy}{y^2} = \int_{\mathcal{G}} (FVG - GFV) \frac{dx dy}{y^2}. \]
(2.18.7)
\[ \text{see Watson, Theory of Bessel functions 4.16.} \]
by Green's Theorem. The integrals along \( x = \pm \frac{1}{2} \) cancel, since \( F(x + 1) = F(x) \), \( G(x + 1) = G(x) \) (see (2.18.13). Similarly the integral for \( |z| = 1 \) vanishes, since \( F(-1/z) = F(z) \), \( G(-1/z) = G(z) \). Thus

\[
\frac{1}{4\pi} \int_{\gamma} F(x, y) \frac{dxdy}{y^2} = -\frac{1}{4} \left( \frac{\partial G}{\partial y}(x, y) - G(x, y) \frac{\partial F}{\partial y}(x, y) \right) dx.
\]

(2.18.8)

The functions \( y^+ \) and \( y^- \) also satisfy the eigenfunction equation (2.18.5) (by (2.18.2) with \( c = 0, d = 1 \)) and thus, by (2.18.4) so too does \( B_2(x, a) \). Consequently, if \( Z > Y \), an argument analogous to that above yields

\[
\frac{1}{4\pi} \int_{\gamma} F(x, y) \frac{dxdy}{y^2} = \frac{1}{4} \left( \frac{\partial G}{\partial y}(x, y) - G(x, y) \frac{\partial F}{\partial y}(x, y) \right) dx
\]

where \( F_x = B_2(x, a) \), \( G_y = B_2(x, w) \). Here we have used \( F_{2}(x + 1) = F_{2}(x) \) and \( G_{2}(x + 1) = G_{2}(x) \). (Note that we no longer have the corresponding relations involving \(-1/z\) when \( Z \to \infty \), using (2.18.6) and (2.18.8), so that the first integral on the right above vanishes. Omitting the result to (2.18.8) we obtain the Massey-Siegel formula

\[
\frac{1}{4\pi} \int_{\gamma} F_{2}(x, a) \frac{dxdy}{y^2} = \frac{1}{4} \left( \frac{\partial G_{2}}{\partial y}(x, y) - G_{2}(x, y) \frac{\partial F_{2}}{\partial y}(x, y) \right) dx.
\]

(2.18.9)

2.19. The Functional Equation

where

\[
\begin{cases}
R(x, a) & (x > Y) \\
B_2(x, a) & (y > Y)
\end{cases}
\]

(2.19.1)

2.18. In the general case there are now various ways in which one can proceed in order to get the analytic continuation of \( \phi \) and \( \psi \). However one point is immediate: once the analytic continuation has been established one may take \( w = 1 - \tau \) in (2.18.9) to obtain the relation

\[
\phi(a) \phi(1 - a) = \psi(a) \psi(1 - a),
\]

(2.19.2)

which can be thought of as a weak form of the functional equation.

The analysis we shall give takes advantage of certain special properties not available in the general case. We shall take \( Y = 1 \) in (2.18.9) and expand the integral on the left to obtain

\[
(1 - w) (a + w) \psi(a) \psi(a) + \beta(a, w) \phi(a) + \gamma(a, w) \phi(a) + \delta(a, w) = 0,
\]

(2.19.2')

where

\[
\alpha(a) = (1 - a) \int_{0}^{a} x^+ x^+ dx - a = \frac{1}{4} \left( 1 - a^2 \right) (1 - a) dx.
\]

(2.19.2)

and \( \beta, \gamma, \delta \) involve the functions \( \phi \) and \( B_2 \), but not \( \psi \). If we know that \( \phi \) has a continuation to the half plane \( \Re(z) > \gamma \), then \( \phi(z) \) has a continuation to \( \Re(z) > \frac{1}{2} \gamma \), so that \( \alpha, \beta, \gamma, \delta \) are meromorphic there. If

\[
(1 - w) \phi(a) + w \psi(a) + \beta(a, w),
\]

(2.19.2)

identically for \( R(x), R(a) > 1 \), then

\[
\psi(a) = \frac{\beta(a, w)}{(a - w) \psi(a)},
\]

(2.19.4)

which gives the analytic continuation of \( \psi(a) \) to \( R(a) > \frac{1}{2} \gamma \). Note that \( (a - w) \psi(a) \) does not vanish identically. If (2.19.3) does not hold for all \( a \) and \( w \) then (2.19.2) yields

\[
\psi(a) = \frac{\beta(a, w) \phi(a) + \delta(a, w)}{(a - w) \phi(a) + \beta(a, w)},
\]

(2.19.5)

which gives the analytic continuation of \( \psi(a) \) to \( R(a) > \frac{1}{2} \gamma \), on choosing a suitable \( w \) in the region \( R(w) > 1 \). In either case \( \phi(a) \) may be continued to \( R(a) > \frac{1}{2} \gamma \). This process shows that \( (a) \) has a meromorphic continuation to the whole complex plane.
Some information on possible poles comes from taking $w = \hat{s}$ in (20.18.5), so that $R(2; \hat{s}) = R(2; s)$. Then

$$\left(1 - \frac{2\pi i}{Y^2} \right) \int_{-\infty}^{\infty} \frac{dx}{x^2} = N^s \left[ \zeta(s) \right] \left[ \frac{1}{2} \right],$$

if $t \neq 0$ we may choose $Y \geq 1$ so that the second term on the right vanishes. It follows that

$$|\zeta(s)| = \frac{|\zeta(2; s)|}{2^s},$$

for $s \geq \frac{1}{2}$. Thus $\zeta$ is regular for $s \geq \frac{1}{2}$ and $t \neq 0$, providing that $\phi$ is. Hence $\zeta(s)$ has no poles for $R(2; s) > 0$, except possibly on the real axis.

If we take $\hat{s} < R(2; s) < 1$, we see that $\zeta(s)$ and $\phi(\hat{s})$ are regular, we see that $\phi(\hat{s})$ can only have a pole at a point $s_0$ for which the denominator vanishes identically in $w$. For such an $s_0$, (20.18.2) must hold. However $\zeta(\hat{s})$ is clearly non-zero for real $\hat{s}$, whence $\phi(\hat{s})$ can have at most a single, simple pole for real $w > \frac{1}{2}$, and this at $w = s_0$. Since it is clear that $\zeta(s)$ does in fact have a singularity at $s = 1$, we see that $s_0 = 1$.

Much of the elegance of the above analysis arises from the fact that, in the general case where one uses the Eisenstein series rather than the Epstein zeta-function, one has a single function $\phi(s) = \phi(\hat{s})$ rather than two separate ones. Here $\hat{s}(w)$ will indeed have poles to the left of $R(2; s) = \frac{1}{2}$.

In our special case we can extract the functional equation for $\zeta(s)$ itself, rather than the weaker relation $\zeta(s) = \zeta(1-s)$ (see (20.18.10)) by using (20.18.7) and (20.18.5). We observe that

$$N^{-1/2} \xi(x, \hat{s}(x)) = n^{-1/2} \xi(x, x),$$

and that $K(x) = K_0(x), \kappa(x)$, where $x^{-1/2} K_0(x, x)$ is invariant under the transformation $s = 1 - s$. It follows that

$$\zeta(s) = \zeta(1-s) = n^{-1/2} \xi(x, x),$$

where we have written temporarily $A(s) = N^{-1/2} (1-s) \xi(x, x)$. The left-hand side is invariant under the transformation $z = -1/\hat{s}$, by (20.18.5), and so, taking $\hat{s} = iy$ for example, we see that $A(s) = A(1-y) = A(1-s)$.

These produce the functional equation in the form (20.6.4) and

$$\zeta(s) = \xi(x, x),$$

indeed yield

$$\zeta(s) = \xi(x, x),$$

where the integral on the right is over the ideals $J$ of $\mathbb{R}$. Here $f$ is in one of a certain class of functions and $c$ is any quasi-character of $\mathbb{R}$ (that is to say, a continuous homomorphism from $\mathbb{R} \rightarrow \mathbb{C}^*$) which is trivial on $k^*$. We may write $c(f)$ in the form $c(f(a))a_1$, where $c(a)$ is a character on $\mathbb{R}$ (i.e. $c(a)(b) = 1$ for $a = b$). Then $c(f)$ corresponds to $\xi$ as a Hecke character for $\mathbb{R}$, and (20.6.4) differs from

$$\zeta(s) = \prod_{f \in \mathbb{R}} \left(1 - \zeta(f) \right) \left(\zeta(f) \right)^{-1},$$

where $P$ runs over prime ideals of $\mathbb{R}$, in only a finite number of factors. In particular, if $k = \mathbb{R}$, then $\zeta(f, c)$ is essentially a Dirichlet $L$ series $\zeta(s, \chi)$. Thus these are essentially the only functions which can be associated to the rational field in this manner.

Tate goes on to prove a Poisson summation formula in this adelic setting, and deduces the elegant functional equation

$$\zeta(f, c) = \zeta(f, c),$$

where $f$ is the 'Fourier transform' of $f$, and $c(f) = c(f(a))a_1$. The functional equation for $(f, c)$ may be extracted from this. In the case $k = \mathbb{R}$ we may take $c(f)$ identically equal to 1, and make a particular choice $f = f_0$, such that $\zeta(f_0) = 1$ and

$$\zeta(f_0, c) = \prod_{f \in \mathbb{R}} \left(1 - \zeta(f) \right) \left(\zeta(f) \right)^{-1},$$

where we have written temporarily $A(s) = N^{-1/2} (1-s) \xi(x, x)$. The left-hand side is invariant under the transformation $z = -1/\hat{s}$, by (20.18.5), and so, taking $\hat{s} = iy$ for example, we see that $A(s) = A(1-y) = A(1-s)$.

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$$\zeta(f_0, c) = \prod_{f \in \mathbb{R}} \left(1 - \zeta(f) \right) \left(\zeta(f) \right)^{-1},$$

where we have written temporarily $A(s) = N^{-1/2} (1-s) \xi(x, x)$. The left-hand side is invariant under the transformation $z = -1/\hat{s}$, by (20.18.5), and so, taking $\hat{s} = iy$ for example, we see that $A(s) = A(1-y) = A(1-s)$.
for \( s \in \mathbb{Z}_p \), (the \( p \)-adic integers) such that
\[
\zeta_p(n) = (1 - p^{-s})^{-1} \text{ for } k \leq 0, k \equiv n \pmod{p-1}.
\]
Indeed if \( n \not\equiv 1 \pmod{p-1} \) then \( \zeta_p(n) \) will be analytic on \( \mathbb{Z}_p \), and if \( n \equiv 1 \pmod{p-1} \) then \( \zeta_p(n) \) will be analytic apart from a simple pole at \( s = 1 \), of residue \( 1 - (1/p) \). These results are due to Leopoldt and Kubota [1]. While these \( p \)-adic zeta-functions seem to have little interest in the simple case above, their generalizations to Dirichlet L-functions yield important algebraic information about the corresponding cyclotomic fields.

III

THEOREM OF HADAMARD AND DE LA VALLEE POUSIN, AND ITS CONSEQUENCES

3.1. As we have already observed, it follows from the formula
\[
\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (s > 1)
\]
that \( \zeta(s) \) has no zeros for \( s > 1 \). For the purpose of prime-number theory, and indeed to determine the general nature of \( \zeta(s) \), it is necessary to extend as far as possible this zero-free region.

It was conjectured by Riemann that all the complex zeros of \( \zeta(s) \) lie on the 'critical line' \( s = \frac{1}{2} \). This conjecture, now known as the Riemann hypothesis, has never been either proved or disproved.

The problem of the zero-free region appears to be a question of extending the sphere of influence of the Euler product (3.1.1) beyond its actual region of convergence; for example, are known of functions which are extremely like the zeta-function in their representation by Dirichlet series, functional equation, and so on, but which have no Euler product, and for which the analogue of the Riemann hypothesis is false. In fact the deepest theorems on the distribution of the zeros of \( \zeta(s) \) are obtained in the way suggested. But the problem of extending the sphere of influence of (3.1.1) to the left of \( s = 1 \) in any effective way appears to be of extreme difficulty.

By (1.1.4)
\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s} \quad (s > 1),
\]
where \( |\mu(n)| \leq 1 \). Hence for \( s \) near to 1
\[
\left| \frac{1}{\zeta(s)} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) < \frac{A}{s-1},
\]
i.e.
\[
|\zeta(s)| > A(s-1).
\]
Hence if \( \zeta(s) \) has a zero on \( s = 1 \) it must be a simple zero. But to prove that there cannot be even simple zeros, a much more subtle argument is required.

It was proved independently by Hadamard and de la Vallée Pousin in 1896 that \( \zeta(s) \) has no zeros on the line \( s = 1 \). Their methods are similar in principle, and they form the main topic of this chapter.
The main object of both these mathematicians was to prove the prime-number theorem, that is, \( n = \sum \frac{1}{\log x} \).

This had previously been conjectured on empirical grounds. It was shown by arguments depending on the theory of functions of a complex variable that the prime-number theorem is a consequence of the Hadamard-de la Vallée Poussin theorem. The proof of the prime-number theorem so obtained was therefore not elementary.

An elementary proof of the prime-number theorem, i.e. a proof not depending on the theory of \( \zeta(s) \) and complex function theory, has recently been obtained by A. Selberg and Erdős. Since the prime-number theorem implies the Hadamard-de la Vallée Poussin theorem, this leads to a new proof of the latter. However, the Selberg-Erdős method does not lead to such good estimates as the Hadamard-de la Vallée Poussin method, so that the latter is still of great interest.

### 3.2. Hadamard's argument is roughly as follows. We have for \( \sigma > 1 \)

\[
\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} - \sum_{n=1}^{p_n} f(n), \quad (3.2.1)
\]

where \( f(n) \) is regular for \( \sigma > 1 \). Since \( \zeta(s) \) has a simple pole at \( s = 1 \), it follows in particular that, as \( \sigma \to 1 \) (\( \sigma > 1 \)),

\[
\sum_{n=1}^{\infty} \frac{1}{p_n^s} \sim \log \frac{1}{(s-1)}, \quad (3.2.2)
\]

Suppose now that \( s = 1 + it \) is a zero of \( \zeta(s) \). Then if \( s = \sigma + it \), as \( \sigma \to 1 \) (\( \sigma > 1 \)),

\[
\sum_{n=1}^{\infty} \frac{\cos(\log p)}{p^s} = \log[\zeta(s)] - R(s) \sim \log(\sigma - 1). \quad (3.2.3)
\]

Comparing (3.2.2) and (3.2.3), we see that \( \cos(\log p) \) must, in some sense, be approximately \( 1 \) for most values of \( p \). But then \( \cos(2\pi \log p) \) is approximately \( 1 \) for most values of \( p \), and

\[
\log[\zeta(s+2\pi i)] = \sum_{n=1}^{\infty} \frac{\cos(2\pi \log p)}{p^s} \sim \sum_{n=1}^{\infty} \frac{1}{p^s} \sim \log \frac{1}{1 - \sigma - 1},
\]

so that \( 1 + 2\pi i \) is a pole of \( \zeta(s) \). Since this is false, it follows that \( \zeta(1 + it) \neq 0 \).

To put the argument in a rigorous form, let

\[
S = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad P = \sum_{n=1}^{p_n} \frac{\cos(\log p)}{p^s}, \quad Q = \sum_{n=1}^{\infty} \frac{\cos(2\pi \log p)}{p^s}.
\]

### 3.3 AND DE LA VALLÉE POUSSIN

Let \( S', P', Q' \) be the parts of these sums for which

\[
(2k+1)x - a \leq \log p \leq (2k+1)x + a
\]

for any integer \( k \), and \( a \) fixed, \( 0 < a < 2\pi \). Let \( S', \) etc., be the remainders. Let \( \lambda = S'/S \).

If \( \varepsilon \) is any positive number, it follows from (3.2.2) and (3.2.3) that

\[
P < -(1-\varepsilon)S
\]

if \( \sigma-1 \) is small enough. But

\[
P' \geq -S' \cos a \geq - \lambda S
\]

and

\[
P' \geq -S' \cos a \geq -(1-\varepsilon)S
\]

Hence

\[
-(1-\varepsilon)S \leq -(1-\varepsilon)S
\]

i.e.

\[
(1-\varepsilon)(1-\cos a) < \varepsilon
\]

Hence \( \lambda \to 1 \) as \( \sigma \to 1 \).

Also

\[
Q' \geq S' \cos 2a, \quad Q' \geq -S'
\]

so that

\[
Q \geq S'(\cos 2a - 1 + \lambda)
\]

Since \( \lambda = 1 \), \( S' \to \infty \), it follows that \( Q \to \infty \) as \( \sigma \to 1 \). Hence \( 1 + 2\pi i \) is a pole, and the result follows as before.

The following form of the argument was suggested by Dr. F. V. Atkinson. We have

\[
\left( \sum_{n=1}^{\infty} \frac{\cos(\log p)}{p^s} \right)^2 = \left( \sum_{n=1}^{\infty} \frac{\cos(\log p)}{p^s} \right)^4 \leq \sum_{n=1}^{\infty} \frac{\cos(\log p)}{p^s} \sum_{n=1}^{p^s} \frac{1}{p^s}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{p^s} \cos(2\pi \log p) \sum_{n=1}^{p^s} \frac{1}{p^s}
\]

i.e.

\[
P^2 \leq \frac{1}{2} (S + Q) S
\]

Suppose now that, for some \( \varepsilon \), \( P \sim \log(\sigma - 1) \). Since \( S \sim \log(1/(\sigma - 1)) \), it follows that, for a given \( \varepsilon \) and \( \sigma - 1 \) small enough,

\[
(1-\varepsilon)P \log \frac{1}{\varepsilon - 1} \leq \frac{1}{2} (1 + e) \log \frac{1}{\varepsilon - 1} + Q \log \frac{1}{\varepsilon - 1},
\]

i.e.

\[
Q \geq \frac{1}{2} (1 + e) \log \frac{1}{\varepsilon - 1}
\]

Hence \( Q \to \infty \), and this involves a contradiction as before.
3.3. In de la Vallée Poussin's argument a relation between $|z| = 1$ and $e^{-n\pi i}$ is also fundamental, but the result is now deduced from the fact that
\[ 3 + 4 \cos \phi + \cos 3\phi = 2(1 + \cos \phi)^2 \geq 0 \] (3.3.1)
for all values of $\phi$.

We have
\[ |z| = \exp \sum_{n=1}^{\infty} \frac{1}{2n^2 \pi^2} \]
and hence
\[ |z| = \exp \sum_{n=1}^{\infty} \frac{\cos (2\pi n \log p)}{mp^n} \]
Hence
\[ P(|z|)^2 = |z|^{2\pi} |z(1+2i)|^{2\pi} \]
\[ = \exp \left( \sum_{p} \sum_{n=1}^{\infty} \frac{3 + 4 \cos (2\pi n \log p) + \cos (2\pi (n+1) \log p)}{mp^n} \right) \] (3.3.2)

Since every term in the last sum is positive or zero, it follows that
\[ P(|z|) |z(1+i)| |z(1+2i)| \geq 1 \quad (\sigma > 1). \] (3.3.3)

Now, keeping $f$ fixed, let $\sigma = 1$. Then
\[ P(|z|) = (1-1)^{\sigma} = 0 \]
and if $1+it$ is a zero of $\zeta(s)$, $|z(1+i)| = 0(t = 0).$ Also $|z(1+2i)| = 0(1)$, since $\zeta(s)$ is regular at $1+2i$. Hence the leading term of (3.3.3) is
\[ |z|^{2\pi} |z(1+2i)| \]
This proves the theorem.

There are other inequalities of the same type as (3.3.1), which can be used for the same purpose, e.g. from
\[ 3 + 8 \cos \phi + 4 \cos 3\phi = (1 + \cos \phi)(1 + \cos 2\phi) \geq 0 \] (3.3.4)
we deduce that
\[ P(|z|) |z(1+i)| |z(1+2i)| \geq 1 \] (3.3.5)
This, however, has no particular advantage over (3.3.3).

3.4. Another alternative proof has been given by Ingham.† This depends on the identity
\[ P(|z|) |z(1+i)| |z(1+2i)| = \sum_{n=1}^{\infty} \frac{|z_n|}{n^{1/2}} \quad (\sigma > 1), \] (3.4.1)
where $\sigma$ is any real number other than zero, and
\[ z_n = \frac{\zeta(n)}{n^{1/2}}. \]
† Ingham (3).
THEOREM OF HADAMARD Chap. III

the result holding by analytic continuation for \( \sigma > 0 \). Hence for \( \sigma > 0 \),
\[ t > 1. \]
\[ \xi(s) - \sum_{n=1}^{\infty} \frac{1}{n^s} = O\left( \int \frac{dx}{x^{s+1}} \right) + O(N^{-s}) \]
\[ = O\left( \frac{t}{\log t} \right) + O\left( \frac{N^{1-s}}{s} \right) + O(N^{-s}). \] (3.5.4)

In the region considered, if \( s \leq t \),
\[ |n-s| = n-s = e^{-s \log n} \approx \exp \left( - \frac{1}{\log n} \right) \log n \leq n^{-s}. \]

Hence, taking \( N = [t] \),
\[ \xi(s) = \sum_{n=1}^{[t]} \frac{1}{n^s} + O\left( \frac{t}{\log t} \right) + O\left( \frac{1}{t} \right) + O\left( \frac{1}{N} \right) \]
\[ = O(\log N) + O(1) = O(\log t). \]

This result will be improved later (Theorem 3.16, §64), but at the
cost of far more difficult proofs.

It is also easy to see that
\[ \xi(s) = O(\log^2 t) \] (3.5.5)
in the above region. For, differentiating (3.3.3),
\[ \zeta' (s) = - s \zeta (s) + \frac{1}{s} \sum \frac{\log n}{n^s} + \int \frac{[s]-1}{s} \left( 1-s \log x \right) dx - \frac{N^{1-s} \log N}{s-1} - \frac{N^{1-s}}{s-1} + \frac{N^{1-s} \log N}{s-1}, \]
and a similar argument holds, with an extra factor \( \log t \) on the right-hand
side. Similarly for higher derivatives of \( \xi(s) \).

We may note in passing that (3.3.3) shows the behaviour of the
Dirichlet series (1.1.1) for \( s \leq 1 \). If we take \( \sigma = 1 \), \( t > 0 \), we obtain
\[ \zeta(1+it) = \sum_{n=1}^{\infty} \frac{1}{n^{1+it}} = \left( 1 + it \right) \sum \frac{\zeta(x+it)}{x^{s+1}} dx - \frac{N^{1-s}}{s-1} + \frac{N^{1-s} \log N}{s-1}, \]
which oscillates finitely as \( N \to \infty \). For \( s < 1 \) the series, of course,
diverges (oscillates infinitely).

3.6. Inequalities for \( \xi(s) \), \( \xi'(s) \), and \( \log \xi(s) \). Inequalities of
this type in the neighbourhood of \( s = 1 \) can now be obtained by a slight
elaboration of the argument of Theorem 3.3. We have for \( \sigma > 1 \)
\[ \left| \frac{1}{\xi(s) - i} \right| \leq \left| \frac{\xi(s)}{\xi(s) - i} \right| \leq O\left( \frac{\log^2 \xi(s)}{\log \xi(s) - 1} \right). \] (3.6.1)
integral formula. We can reduce this to a case of Mellin’s inversion formula as follows. Let

\[ \omega(x) = \int \frac{\pi(x)}{x^{1+\sigma} - 1} \, dx. \]

Then

\[ \log \frac{\pi(x)}{x} - \omega(x) = -\int \frac{\pi(x)}{x^{1+\sigma}} \, dx. \] (3.7.2)

This is of the Mellin form, and \( \omega(x) \) is a comparatively trivial function; in fact since \( \pi(x) \ll x \), the integral for \( \omega(x) \) converges uniformly for \( \sigma > \frac{1}{2} + \delta \), by comparison with

\[ \int \frac{1}{x^{1+\delta}(x^{1+\delta} - 1)} \, dx. \]

Hence \( \omega(x) \) is regular and bounded for \( \sigma > \frac{1}{2} + \delta \). Similarly so is \( \omega'(x) \), since

\[ \omega'(x) = \int \frac{\pi(x) \log x}{x^{1+\sigma}(x^{1+\sigma} - 1)} \, dx. \]

We could now use Mellin’s inversion formula, but the resulting formula is not easily manageable. We therefore modify (3.7.2) as follows. Differentiating with respect to \( x \),

\[ -\int \frac{\pi(x)}{x^{1+\delta}(x^{1+\delta} - 1)} \, dx + \omega'(x) = -\int \frac{\pi(x) \log x}{x^{1+\delta}} \, dx. \]

Denote the left-hand side by \( \phi(\delta) \), and let

\[ g(x) = \int \frac{\pi(u) \log u}{u} \, du, \quad h(x) = \int \frac{\pi(u)}{u} \, du. \]

\( \nu(x) \), \( \psi(x) \), and \( \delta(x) \) being zero for \( x < 2 \). Then, integrating by parts,

\[ \phi(\delta) = -\int \frac{\pi(x) \log x}{x^{1+\delta}} \, dx = \int \frac{\psi(x)}{x^{1+\delta}} \, dx \]

\[ = -\int \frac{\pi(x) \log x}{x^{1+\delta}} \, dx = \delta \int \frac{\pi(x) \log x}{x^{1+\delta}} \, dx \quad (\sigma > 1), \]

or

\[ \delta(1-\delta) \int \frac{\pi(x)}{x^{1+\delta}} \, dx = \frac{\pi(x)}{x^{1+\delta}} \, dx. \]

Now \( h(x) \) is continuous and of bounded variation in any finite interval; and\(, \) since \( \nu(x) \ll x \), it follows that, for \( x \gg 1 \), \( \psi(x) \ll \log x \), and \( h(x) \ll x \log x \). Hence \( k(x)x^{\alpha-1} \) is absolutely integrable over \((0, \infty)\) if \( k < 0 \), hence

\[ k(x) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\phi(s)}{s^{\sigma+1}} \, ds \quad (k < 0), \]

or

\[ h(x) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\psi(s)}{s^{\sigma}} \, ds \quad (c > 1). \]

The integral on the right is absolutely convergent, since by (3.6.6) and (3.6.7) \( \phi(s) \) is bounded for \( s > 1 \), except in the neighbourhood of \( s = 1 \).

In the neighbourhood of \( s = 1 \)

\[ \phi(s) = \frac{1}{s-1} + \log \frac{1}{s-1} + \ldots, \]

and we may write

\[ \phi(s) = \frac{1}{s-1} + \psi(s), \]

where \( \psi(s) \) is bounded for \( s > 1 \), \( |s-1| > 1 \), and \( \psi(s) \) has a logarithmic infinity as \( s \to 1 \). Now

\[ h(x) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\pi(x)}{x^{1+\sigma}} \, ds + \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \frac{\pi(x)}{x^{1+\sigma}} \, ds. \]

The disc \( |s| = 1 \) is exterior to the sum of the residues on the left of the line \( \Re(s) = c \), and so is

\[ x - \log x - 1. \]

In the other term we may put \( c = 1 \), i.e. apply Cauchy’s theorem to the rectangle \((1 \pm \delta i, e \pm 2 \delta i)\), with an indentation of radius \( \varepsilon \) round \( s = 1 \), and make \( T \to \infty, \varepsilon \to 0 \). Hence

\[ h(x) = x - \log x - 1 + \int_{-\infty}^{\infty} \frac{\pi(x)}{x^{1+\sigma}} \, ds. \]

The last integral tends to zero as \( x \to \infty \), by the extension to Fourier integrals of the Riemann-Lebesgue theorem.\(^*\) Hence

\[ h(x) \sim x. \] (3.7.3)

\(^*\) See my Introduction to the Theory of Fourier Integrals, Theorem 1.
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To get back to $(e)_{x}$ we now use the following lemma:
Let $f$ be positive non-decreasing, and as $x \to x_{0}$ let

$$
\int_{x_{0}}^{x} \frac{f(u) \, du}{u} \sim x.
$$

Then

$$
f(x) \sim x.
$$

If $\delta$ is a given positive number,

$$
(1-\delta)x < \int_{x_{0}}^{x} f(t) \, dt < (1+\delta)x \quad (x > x_{0}(\delta)).
$$

Hence for any positive $\epsilon$

$$
\int_{x}^{x+\epsilon} \frac{f(u) \, du}{u} \sim \int_{x}^{x+\epsilon} f(u) \, du \sim \int_{x}^{x+\epsilon} f(u) \, du
$$

$\leq (1+\delta)(1+\epsilon)x - (1-\delta)x

= (2\delta + \epsilon + 3) \epsilon x$.

But, since $f(x)$ is non-decreasing,

$$
\int_{x}^{x+\epsilon} \frac{f(u) \, du}{u} \geq \int_{x}^{x+\epsilon} f(x) \, du \sim \int_{x}^{x+\epsilon} \frac{f(x) \, du}{x(1+\epsilon)} = \frac{\epsilon}{1+\epsilon} f(x).
$$

Hence

$$
f(x) \sim x(1+\epsilon) \left(1+\delta + \frac{2\delta}{x} \right).
$$

Taking, for example, $\epsilon = \sqrt{x}$, it follows that

$$
\lim_{x \to \infty} \frac{f(x)}{x} < 1.
$$

Similarly, by considering

$$
\lim_{x \to \infty} \frac{\int_{x}^{x+\epsilon} \frac{f(u) \, du}{u}}{\int_{x}^{x+\epsilon} \frac{f(u) \, du}{u}} = 1,
$$

we obtain

$$
\lim_{x \to \infty} \frac{f(x)}{x} \geq 1,
$$

and the lemma follows.

Applying the lemma twice, we deduce from (3.7.3) that

$$
g(x) \sim x,
$$

and hence that

3.8. THEOREM 3.8. There is a constant $A$ such that $\xi(x)$ is not zero for

$$
s > 1 - \frac{A}{\log t} \quad (t > t_{0}).
$$

We have for $s > 1$

$$
-R \left[ \frac{\zeta'(s)}{\zeta(s)} \right] = \sum_{p} \frac{\log p}{p^{s}} \cos \left( \pi \log p \right).
$$

Hence, for $s > 1$ and any real $\gamma$,

$$
-\frac{\zeta'(s)}{\zeta(s)} = \text{Re} \left[ \frac{\zeta'(s+i\gamma)}{\zeta(s+i\gamma)} \right] - \text{Re} \left[ \frac{\zeta'(s+2i\gamma)}{\zeta(s+2i\gamma)} \right]

= \sum_{p < 2^{x}} \frac{\log p}{p^{s}} \left( 3 + 4 \cos(\gamma \log p) + \cos(2\gamma \log p) \right) \geq 0.
$$

Now

$$
-\frac{\zeta'(s)}{\zeta(s)} < \frac{1}{s-1} + O(1).
$$

Also, by (2.12.7),

$$
-\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} - \frac{1}{s-\rho - 1} \right)
$$

where $\rho = \beta + iy$ runs through complex zeros of $\zeta(s)$. Hence

$$
-R \left[ \frac{\zeta'(s)}{\zeta(s)} \right] = O(\log t) - \sum_{\rho} \left( \frac{s-\beta}{(s-\beta)^{2} + (1-\beta)^{2} + (1/\beta - 1/\beta)^{2}} \right)
$$

Since every term in the last sum is positive, it follows that

$$
-R \left[ \frac{\zeta'(s)}{\zeta(s)} \right] < O(\log t),
$$

and also, if $\rho_{0} + iy$ is a particular zero of $\zeta(s)$, that

$$
-R \left[ \frac{\zeta'(s)}{\zeta(s)} \right] < O(\log y) - \frac{1}{\sigma - \beta}.
$$

From (3.8.2), (3.8.3), (3.8.5), (3.8.6) we obtain

$$
\frac{3}{s-\sigma} - \frac{4}{\sigma - \beta} - O(\log y) > 0,
$$

or say

$$
\frac{3}{s-\sigma} - \frac{4}{\sigma - \beta} > -A_{\sigma} \log y.
$$

Solving for $\sigma$, we obtain

$$
1 - \beta > \frac{1 - A_{\sigma}}{\log y}.
$$

The right-hand side is positive if $s > 1 - A_{\sigma} / \log y$, and then

$$
1 - \beta > \frac{A_{\sigma}}{\log y},
$$

The required result.
3.9. There is an alternative method, due to Landau,† of obtaining results of this kind, in which the analytic character of \(\zeta(s)\) for \(s < 0\) need not be known. It depends on the following lemmas.

**Lemma a.** If \(f(s)\) is regular, and
\[
\frac{f'(s) - f(s)}{f(s)} < e^M \quad (M > 1)
\]
in the circle \(|s - \eta| < r\), then
\[
\left| \frac{f'(s)}{f(s)} + \sum_{\rho} \frac{1}{s - \rho} - \frac{A M}{r} \right| < \frac{A M}{r} \left| s - \eta \right| \leq r
\]
where \(\rho\) runs through the zeros of \(f(s)\) such that \(|s - \eta| < 4r\).

The function \(g(s) = f(s) \prod (s - \rho)^{-1}\) is regular for \(|s - \eta| < r\), and not zero for \(|s - \eta| < 4r\). On \(|s - \eta| = r\), \(|s - \rho| \geq \frac{1}{r} \geq |s - \eta - \rho|\), so that
\[
\left| \frac{g'(s)}{g(s)} \right| = \left| \frac{f'(s)}{f(s)} \prod (s - \rho)^{-1} \right| \leq \left| \frac{f'(s)}{f(s)} \right| < e^M.
\]
This inequality therefore holds inside the circle also. Hence the function
\[
h(s) = \log \left( \frac{g(s)}{g(\eta)} \right)
\]
where the logarithm is zero at \(s = \eta\), is regular for \(|s - \eta| < 4r\), and
\[h(\eta) = 0, \quad \mathfrak{R}(h(s)) < M.
\]
Hence by the Borel–Casorati theorem
\[
|\mathfrak{R}(h(s)| < 4M \left| s - \eta \right| < 4r
\]
and so, for \(|s - \eta| < 4r\),
\[
|\mathfrak{R}(h(s)| = \frac{1}{2\pi i} \int_{|s - \eta - \rho| = 4r} \frac{h(s)}{(s - \rho)^2} ds < \frac{AM}{r}.
\]
This gives the result stated.

**Lemma b.** If \(f(s)\) satisfies the conditions of the previous lemma, and has no zeros in the right-hand half of the circle \(|s - \eta| \leq r\), then
\[
-\mathfrak{R} \left( \frac{f'(s)}{f(s)} \right) < \frac{AM}{r}
\]
while if \(f(s)\) has a zero \(\eta\) between \(s = \eta - 4r\) and \(s = \eta\), then
\[
-\mathfrak{R} \left( \frac{f'(s)}{f(s)} \right) < \frac{AM}{r} = \frac{1}{s - \eta - p}.
\]

† Landau (14). † Titchmarsh, Theory of Functions, § 5.5.

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3.10. We can now prove the following general theorem, which we shall apply later with special forms of the functions \(g(t)\) and \(\phi(t)\).

**Theorem 3.10.** Let \(\zeta(s) = g(t) e^{\phi(t)}\) as \(t \to \infty\) in the region
\[
1 - \theta(t) < \alpha < 2 \quad (t > 0),
\]
where \(\phi(t)\) and \(1/\theta(t)\) are positive non-decreasing functions of \(t\) for \(t > 0\), such that \(\theta(t) \leq 1\), \(\phi(t) \to \infty\), and
\[
\frac{\phi(t)}{\theta(t)} = o(\theta(t)).
\]
Then there is a constant \(A_1\) such that \(\zeta(s)\) has no zeros in the region
\[
s \geq 1 - A_1 \theta(t) e^{\phi(t)}
\]
Let \(\beta + iy\) be a zero of \(\zeta(s)\) in the upper half-plane. Let
\[
1 + e^{\phi(t)} \leq e^{\delta_0} \leq 2,
\]
\[
\delta_0 = \alpha_0 + \delta_0, \quad \alpha_0 = 2\delta_0, \quad r = \theta(t) + 1.
\]

**Lemma a** gives
\[
\mathfrak{R} \left( \frac{f'(s)}{f(s)} \right) < \frac{AM}{r} - \sum_{\rho} \frac{1}{s - \rho},
\]
and since \(\mathfrak{R}(1/\theta(t)) \geq 0\) for every \(\rho\), both results follow at once.

**Lemma b.** Let \(f(s)\) satisfies the conditions of Lemma a, and let
\[
\left| \frac{f'(s)}{f(s)} \right| < \frac{M}{r}.
\]
Suppose also that \(f(s) \neq 0\) in the part \(\alpha > \alpha_0 - 2\delta\) of the circle \(|s - \eta| \leq r\), where \(0 < r' < r\). Then
\[
\left| \frac{f'(s)}{f(s)} \right| < \frac{AM}{r} \left| s - \eta \right| \leq r'.
\]

**Lemma a** now gives
\[
-\mathfrak{R} \left( \frac{f'(s)}{f(s)} \right) < \frac{AM}{r} - \sum_{\rho} \frac{1}{s - \rho} < \frac{AM}{r}
\]
for all \(s \in [\alpha - \eta] \leq r\), \(\alpha \geq \alpha_0 - 2\delta\), each term of the sum being positive in this region. The result then follows on applying the Borel–Casorati theorem to the function \(-f'(s)/f(s)\) and the circles \(|s - \eta| = 2r', \left| s - \eta - \rho \right| = r'\).

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\[
\frac{\phi(t)}{\theta(t)} = o(\theta(t)).
\]
Then there is a constant \(A_1\) such that \(\zeta(s)\) has no zeros in the region
\[
s \geq 1 - A_1 \theta(t) e^{\phi(t)}
\]
Let \(\beta + iy\) be a zero of \(\zeta(s)\) in the upper half-plane. Let
\[
1 + e^{\phi(t)} \leq e^{\delta_0} \leq 2,
\]
where \(\delta_0 = \alpha_0 + \delta_0, \quad \alpha_0 = 2\delta_0, \quad r = \theta(t) + 1.
\]

**Lemma a** gives
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\]
and since \(\mathfrak{R}(1/\theta(t)) \geq 0\) for every \(\rho\), both results follow at once.

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\[
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\]
Suppose also that \(f(s) \neq 0\) in the part \(\alpha > \alpha_0 - 2\delta\) of the circle \(|s - \eta| \leq r\), where \(0 < r' < r\). Then
\[
\left| \frac{f'(s)}{f(s)} \right| < \frac{AM}{r} \left| s - \eta \right| \leq r'.
\]

**Lemma a** now gives
\[
-\mathfrak{R} \left( \frac{f'(s)}{f(s)} \right) < \frac{AM}{r} - \sum_{\rho} \frac{1}{s - \rho} < \frac{AM}{r}
\]
for all \(s \in [\alpha - \eta] \leq r\), \(\alpha \geq \alpha_0 - 2\delta\), each term of the sum being positive in this region. The result then follows on applying the Borel–Casorati theorem to the function \(-f'(s)/f(s)\) and the circles \(|s - \eta| = 2r', \left| s - \eta - \rho \right| = r'\).

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3.10. We can now prove the following general theorem, which we shall apply later with special forms of the functions \(g(t)\) and \(\phi(t)\).

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\[
\frac{\phi(t)}{\theta(t)} = o(\theta(t)).
\]
Then there is a constant \(A_1\) such that \(\zeta(s)\) has no zeros in the region
\[
s \geq 1 - A_1 \theta(t) e^{\phi(t)}
\]
Let \(\beta + iy\) be a zero of \(\zeta(s)\) in the upper half-plane. Let
\[
1 + e^{\phi(t)} \leq e^{\delta_0} \leq 2,
\]
where \(\delta_0 = \alpha_0 + \delta_0, \quad \alpha_0 = 2\delta_0, \quad r = \theta(t) + 1\).
Then the circles \(|s - s_0| \leq r\), \(|s - s_1| \leq r\) both lie in the region
\[\sigma \geq 1 - \theta(t)\]
Now
\[\left| \frac{1}{\zeta(s_0)} - \frac{A}{\zeta(s_1)} \right| < A e^{\sigma \theta(t)}\]
and similarly for \(s_1\). Hence there is a constant \(A_2\) such that
\[\left| \frac{\zeta(s_0)}{\zeta(s_1)} \right| < \frac{A_2 e^{\sigma \theta(t)}}{\theta(t)}\]
in the circles \(|s - s_0| < r\), \(|s - s_1| < r\) respectively. We can therefore apply Lemma \(\beta\) with \(M = A_3 \theta(2y + 1)\). We obtain
\[-R\left(\zeta'(s_0 + 2y)\right) < A_3 \frac{\theta(2y + 1)}{\theta(2y + 1)} \frac{1}{\zeta(s_0 + 2y)}\]
and, if
\[\beta > \sigma - \delta\]
\[-R\left(\zeta'(s_1 + 2y)\right) < A_3 \frac{\theta(2y + 1)}{\theta(2y + 1)} \frac{1}{\zeta(s_1 + 2y)}\]
Also as \(s_0 \to 1\)
\[\frac{\zeta'(s_0)}{\zeta(s_0)} \sim \frac{1}{\zeta(s_0)} \sim \frac{1}{\zeta(s_1)} \sim \frac{a}{\zeta(s_1)}
\]
Hence
\[\frac{\zeta'(s_0)}{\zeta(s_0)} > \frac{a}{\zeta(s_1)}\]
where \(a\) can be made as near \(1\) as we please by choice of \(s_0\).
Now \((3.8.2), (3.10.3), (3.10.5), (3.10.6)\) give
\[\frac{3a}{\theta(s_1 - 1)} - \frac{3a}{\theta(s_1 - 1)} < \frac{\theta(2y + 1)}{\theta(2y + 1)} \left(\frac{1}{\zeta(s_1 + 2y)} - \frac{1}{\zeta(s_0 + 2y)}\right)\]
where
\[\frac{a}{\zeta(s_1)} \sim a - \frac{\theta(2y + 1)}{\theta(2y + 1)}\]
To make the numerator positive, take \(a = \frac{1}{2}\) and
\[\frac{1}{\zeta(s_0)} = \frac{1}{\theta(2y + 1)} \frac{3a}{\theta(2y + 1)} \frac{\theta(2y + 1)}{\theta(2y + 1)}\]
this being consistent with the previous conditions, by \((3.10.1), \gamma \varphi\) large enough. It follows that
\[1 - \beta > \frac{\theta(2y + 1)}{124a \theta(2y + 1)}\]

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as required. If \((3.10.4)\) is not satisfied,
\[\beta < \sigma - \delta = 1 + \frac{1}{\theta(2y + 1)} \frac{\theta(2y + 1)}{\theta(2y + 1)} \frac{1}{\theta(2y + 1)}\]
which also leads to \((3.10.2)\). This proves the theorem.
In particular, we can take \(\theta(t) = \frac{1}{2}, \delta(t) = -\log(\delta + 2)\). This gives a new proof of Theorem 3.8.

311 THEOREM 3.11. Under the hypotheses of Theorem 3.10 we have
\[\frac{\zeta'(s_0)}{\zeta(s_0)} = O\left(\frac{\theta(2y + 1)}{\theta(2y + 1)}\right) \left(\frac{1}{\zeta(s_0)}\right) \left(\frac{1}{\zeta(s_1)}\right)\]
uniformly for
\[\sigma > 1 - A_3 \theta(2y + 1)\]
In particular
\[\frac{\zeta'(s_0)}{\zeta(s_0)} = O\left(\frac{\theta(2y + 1)}{\theta(2y + 1)}\right) \left(\frac{1}{\zeta(s_0)}\right) \left(\frac{1}{\zeta(s_1)}\right)\]
We apply Lemma \(\gamma\), with
\[\sigma - \delta = 1 + \frac{1}{\theta(2y + 1)} \zeta(s_0), \quad \delta = -\frac{1}{\theta(2y + 1)}\]
In the circle \(|s - s_0| < r\)
\[\frac{\zeta'(s_0)}{\zeta(s_0)} = O\left(\frac{\theta(2y + 1)}{\theta(2y + 1)}\right) \left(\frac{1}{\zeta(s_0)}\right) \left(\frac{1}{\zeta(s_1)}\right)\]
and
\[\frac{\zeta'(s_0)}{\zeta(s_0)} = O\left(\frac{\theta(2y + 1)}{\theta(2y + 1)}\right) \left(\frac{1}{\zeta(s_0)}\right) \left(\frac{1}{\zeta(s_1)}\right)\]
We can therefore take \(M = A \phi(2y + 3)\). Also, by the previous theorem, \(\gamma\) has no zeros for
\[t < t_0 + 1, \quad \sigma > 1 - A_3 \theta(2y + 1) = 1 - A_3 \phi(2y + 3)\]
Hence we can take
\[\zeta'(s_0) = O\left(\frac{\theta(2y + 1)}{\theta(2y + 1)}\right) \left(\frac{1}{\zeta(s_0)}\right) \left(\frac{1}{\zeta(s_1)}\right)\]
for
\[|s - s_0| < \frac{3A_3 \theta(2y + 3)}{4 \theta(2y + 3)}\]
and in particular for
\[t = t_0, \quad \sigma > 1 - A_3 \phi(2y + 3)\]
This is \((3.11.1)\), with \(t_0\) instead of \(t\).
Also, if \( 1 - \frac{A}{g(2\pi + 3)} \leq \sigma \leq 1 + \frac{R(2\pi + 3)}{g(2\pi + 3)} \),
\[
\log \frac{1}{|z(t)|} = -R \log \frac{1}{z(t)} \leq \log \frac{1}{1 + \frac{R(2\pi + 3)}{g(2\pi + 3)}} + \int_1^{rac{1}{1 + \frac{R(2\pi + 3)}}} R \frac{G(u+it)}{G(u)} \mathrm{d}u \leq \log \frac{1}{1 + \frac{R(2\pi + 3)}{g(2\pi + 3)}} + \int_1^{rac{1}{1 + \frac{R(2\pi + 3)}}} \left( \frac{G(2\pi + 3)}{G(2\pi + 3)} \right) \mathrm{d}u \leq \log \frac{1}{1 + \frac{R(2\pi + 3)}{g(2\pi + 3)}} + O(1).
\]
Hence (3.11.2) follows if \( \sigma \) is in the range (3.11.6), and for larger \( \sigma \) it is trivial.

Since we may take \( \theta(t) = 1, \lambda(t) = \log g(t + 2) \), it follows that
\[
\frac{\zeta'(\sigma)}{\zeta(\sigma)} = O(\log t), \quad \frac{1}{\xi(t)} = O(\log t). \tag{3.11.7, 3.11.8}
\]
in a region \( \sigma \geq 1 - A/\log t \), and in particular
\[
\frac{\zeta'(1+it)}{\zeta(1+it)} = O(\log t), \quad \frac{1}{\xi(1+it)} = O(\log t). \tag{3.11.9, 3.11.10}
\]

3.12. For this subsection we require the following lemmas.

**Lemma 3.12.** Let
\[
f(t) = \sum_{n=1}^{\infty} \frac{a_n}{n^t} \quad (\sigma > 1),
\]
where \( a_n = O(n^{\rho(n)}) \), \( \rho(n) \) being non-decreasing, and
\[
\left( \lambda_n \right)_{n=1} = O(1), \quad \left( \frac{\lambda_n}{n^t} \right)_{n=1} = O(1).
\]
as \( t \to 1 \). Then, if \( \epsilon > 0, a_0 > 1, x \) is not an integer, and \( N \) is the integer nearest to 1.

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^t} - \frac{1}{2\pi i} \int_{1-\epsilon}^{1+\epsilon} f(x^2 + w) \mathrm{d}w \leq \frac{O \left( \frac{x^{2\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right)}{1} + O \left( \frac{\lambda_n^{1-\epsilon}}{1} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right) \]
\[
+ O \left( \frac{h^{1-\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right). \tag{3.12.1}
\]

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If \( x \) is an integer, the corresponding result is
\[
\sum_{n=1}^{\infty} \frac{a_n}{n^t} - \frac{1}{2\pi i} \int_{1-\epsilon}^{1+\epsilon} f(x + w) \mathrm{d}w \leq \frac{O \left( \frac{x^{2\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right)}{1} + O \left( \frac{h^{1-\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right).
\]

Suppose first that \( x \) is not an integer. If \( x < x \), the calculus of residues gives
\[
\int_{-\epsilon}^{\epsilon} \frac{f(x^2 + w)}{1} \mathrm{d}w = 2.
\]
Now
\[
\int_{-\epsilon}^{\epsilon} f(x^2 + w) \mathrm{d}w = \int_{-\epsilon}^{\epsilon} f(x^2 + w) \mathrm{d}w + \int_{-\epsilon}^{\epsilon} f(x^2 + w) \mathrm{d}w
\]
\[
= O \left( \frac{x^{2\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right) + O \left( \frac{h^{1-\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right).
\]

and similarly for the integral over \( (-\epsilon, -\epsilon) \) and \( (\epsilon, \epsilon) \) Hence
\[
\int_{-\epsilon}^{\epsilon} f(x^2 + w) \mathrm{d}w = 1 + O \left( \frac{x^{2\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right).
\]

If \( x > x \) we argue similarly with \( -\epsilon \) replaced by \( +\epsilon \), and there is no residue term. We therefore obtain a similar result without the term 1.

Multiplying by \( a_n \) and summing,
\[
\sum_{n=1}^{\infty} \frac{a_n}{n^t} - \frac{1}{2\pi i} \int_{1-\epsilon}^{1+\epsilon} f(x + w) \mathrm{d}w \leq \sum_{n=1}^{\infty} \frac{a_n}{n^t} + O \left( \frac{x^{2\epsilon}}{(2\pi + 3)^{1-\epsilon}} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right).
\]

If \( n < x \) or \( n > 2x \), \( G(N) > A \), and these parts of the sum are
\[
O \left( \frac{\lambda_n}{n^t} \right) + O \left( \frac{G(N)^{1-\epsilon}}{1} \right).
\]

If \( N < n \leq 2x \), let \( n = N + \tau \). Then
\[
\log \frac{n}{x} = \log \frac{N + \tau}{n} > \frac{Ar}{x}.
\]
Hence this part of the sum is
\[ O\left(\varepsilon x^2 \log x \sum \frac{1}{(\varepsilon x^2)^s}\right) = O\left(\varepsilon x^2 \log x + \varepsilon \log x\right). \]
A similar argument applies to the terms with \( x < \alpha < N \). Finally
\[ \frac{1}{N^{x+r} \log z / N!} = O\left(\frac{\varepsilon(N)^2}{N^{x+r} \log(1+(z-N)/N)}\right) = O\left(\frac{\varepsilon(N)^2}{z-N}\right). \]
Hence (3.12.1) follows.
If \( x \) is an integer, all goes so before except for the term
\[ \frac{1}{2(2+i\pi)} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{d\zeta}{\zeta} = \frac{\varepsilon}{2\pi i} \log \left(\frac{1}{x+1}\right) = \text{Residue at } z = x+1, \]
Hence (3.12.2) follows.

3.13. **Theorem 3.13.** We have
\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\rho(n)}{n^s}, \]
at all points of the line \( \sigma = 1 \).

Take \( n = \rho(n), s = 1, w = 1 \), in the lemma, and let \( x \) be half an odd integer. We obtain
\[ \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{(t+iw)} x^t dw + O\left(\frac{\varepsilon}{(t+iw)} x^t \right) + O\left(\frac{\varepsilon}{(t+iw)} x^t \right). \]
The theorem of residues gives
\[ \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{(t+iw)} x^t dw = \frac{\varepsilon}{2\pi i} + \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{(t+iw)} - \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \]
if \( \delta \) is so small that \( (t+iw) \) has no zeros for
\[ R(t) \geq -\delta, \quad |t+iw| \leq |t| + T. \]
By § 3.6 we can take \( \delta = A \log x + T \). Then
\[ \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{(t+iw)} x^t dw = O\left(x^\delta \log x T \int_{-\varepsilon/2}^{\varepsilon/2} \frac{dx}{x^{\delta}}\right) \]
\[ = O\left(x^\delta \log x T \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{(1+iw)}\right) \approx O\left(x^\delta \log x T \right), \]
and similarly for the other integral. Hence
\[ \sum_{n=1}^{\infty} \frac{1}{\zeta(n)} = O\left(\frac{\log x}{(1+iw)} x^t dw + O\left(\frac{\varepsilon}{(t+iw)} x^t \right) + O\left(\frac{\varepsilon}{(t+iw)} x^t \right). \]
Take \( \varepsilon = 1/\log x \), so that \( \varepsilon = 1/\log x \), and take \( T = \exp(\log x)^{2n^2} \), so that \( \log T = (\log x)^{2n^2} \), \( \delta = A(\log x)^{2n^2}, x^2 = T^2 \). Then the right-hand side tends to zero, and the result follows.

In particular
\[ \sum_{n \leq x} \frac{\rho(n)}{n} = 0. \]

3.14. **The series for \( \zeta(n)/\zeta(n) \) and \( \log \zeta(n) \) on \( \sigma = 1 \).**
Taking \( a_n = \Lambda(n) = O(\log n), a = 1, v = 1, \) in the lemma, we obtain
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} - \frac{1}{2\pi i} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{\zeta(t + a) x^t}{(t+a)^2} dw = O\left(\frac{\varepsilon}{(t+a)^2} x^t \right) + O\left(\frac{\varepsilon}{(t+a)^2} x^t \right). \]
In this case there is a pole at \( w = 1 - \sigma \), giving a residue term
\[ \zeta(t) x^\sigma = \left(x^\sigma \right) \quad (\sigma = 1), \quad \log x \quad (\sigma = 1), \]
where \( a \) is a constant. Hence if \( \sigma = 1 \) we obtain
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} + \frac{\zeta(t)}{\zeta(t)} x^\sigma = O\left(\frac{\varepsilon}{(t+a)^2} x^t \right) + O\left(\frac{\varepsilon}{(t+a)^2} x^t \right) + O\left(\frac{\varepsilon}{(t+a)^2} x^t \right) \]
Taking \( \varepsilon = 1/\log x \), \( T = \exp(\log x)^{2n^2} \), we obtain as before
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} + \frac{\zeta(t)}{\zeta(t)} x^\sigma = O\left(\frac{\varepsilon}{(t+a)^2} x^t \right) \quad (\sigma = 1 - \alpha). \]
The term \( x^\sigma/(1-\sigma) \) oscillates finitely, so that if \( R(t) = 1, \sigma = 1 \), the series \( \sum \Lambda(n) x^n \) is not convergent, but its partial sums are bounded.

If \( x = 1 \), we obtain
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} = \log x + O(1), \]
or, since
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} = \sum_{n \geq 2} \frac{\log p}{p} \sum_{n \geq \alpha} \frac{\log p}{p} = \sum_{n \geq \alpha} \frac{\log p}{p} + O(1), \]
and
\[ \sum_{n \geq \alpha} \frac{\log p}{p} = \log x + O(1) \quad (\sigma = 1 - \alpha). \]
Since $\Lambda(t) = \Lambda(t)/\log n$, and $1/\log n$ tends steadily to zero, it follows that
\[ \sum_{x<n} \Lambda(n) \]
is convergent on $\sigma = 1$, except for $t = 0$. Hence, by the continuity theorem for Dirichlet series, the equation
\[ \log \zeta(s) = \sum_{x<n} \Lambda(n) \]
holds for $\sigma = 1$, $t \neq 0$.

To determine the behaviour of this series for $s = 1$ we have, as in the case of $1/\zeta(t)$,
\[ \sum_{x<n} \Lambda(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \log \zeta(w+1) \frac{s^w}{w} \, dw + O\left( \frac{\log x}{T} \right) \]
where $c = 1/\log x$, and $T$ is chosen as before. Now,
\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \log \zeta(w+1) \frac{s^w}{w} \, dw = \frac{1}{2\pi i} \left( \int_{c-iT}^{c+iT} + \int_{-\infty}^{c-iT} - \int_{-\infty}^{c+iT} \right) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \int_{-\infty}^{c+iT} + \int_{c-iT}^{c+iT} \right) \]
where $C$ is a loop starting and finishing at $s = s$, and encircling the origin in the positive direction. Defining $\delta$ as before, the integral along $\sigma = 1/2$ is $O(x^{1/2} \log^2 T)$, and the integrals along the horizontal sides are $O(x^{1/2} \log^2 T)$, by (3.6.7). Since
\[ \frac{1}{\zeta(1+it)} \frac{s^w}{w} \]
is regular at the origin, the last term is equal to
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \log \zeta(w+1) \frac{s^w}{w} \, dw. \]
Since
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \log \zeta(w+1) \frac{s^w}{w} \, dw = -\frac{1}{4\pi i} \Delta_0 \log^2 n \]
\[ = -\frac{1}{4\pi i} \left( \log^2 (\pi x) - 2 \log (\pi e^{\gamma} x) \right) - \log \delta, \]
this term is also equal to
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \log \zeta(w+1) \frac{s^w}{w} \, dw - \log \delta. \]
Take $C$ to be a circle with centre $w = 0$ and radius $\rho (\rho < \delta)$, together with the segment $(-\delta, -\rho)$ of the real axis described twice. The integrals along the real segments together give
\[ \frac{1}{2\pi i} \int_{-\delta}^{-\rho} \log \left( \frac{1}{w+c} \right) x^{w-1} dw - \frac{1}{2\pi i} \int_{-\rho}^{-\delta} \log \left( \frac{1}{w+c} \right) x^{w-1} dw \]
\[ = \int_{-\delta}^{-\rho} \frac{x^{w-1}}{w} dw - \int_{-\delta}^{-\rho} \frac{x^{w-1}}{w} dw \]
\[ = \int_{-\delta}^{-\rho} \frac{1-x^w}{w} dw + \log(\delta \log x) \]
\[ = \gamma + \log(\delta \log x) + o(1) \]
if $\rho \log x \to 0$ and $\delta \log x \to \infty$. Also
\[ \int_{-\delta}^{-\rho} \frac{1-x^w}{w} dw = O\left( \frac{\log x}{\rho} \right) \]
Taking $\rho = 1/\log x$, say, it follows that
\[ \sum_{x<n} \Lambda(x) = \frac{1}{\zeta(1)} \log x + \gamma + O(1). \]
(3.14.4)

The left-hand side can also be written in the form
\[ \sum_{p \leq x} \frac{1}{\log p} + \sum_{x<p} \frac{1}{\log p}. \]
As $x \to \infty$, the second term eventually tends to the limit
\[ \sum_{p \leq x} \frac{1}{\log p}. \]
Hence
\[ \sum_{p \leq x} \frac{1}{p} = \log x + \gamma - \sum_{p \leq x} \frac{1}{\log p} + o(1). \]
(3.14.5)

3.15 Euler's product on $\sigma = 1$. The above analysis shows that for $\sigma = 1$, $t \neq 0$,
\[ \log \zeta(s) = \sum_{p \leq x} \frac{1}{p} + \sum_{x<p} \frac{1}{\zeta(1)} \]
where $p$ runs through primes and $y$ through powers of primes. In fact the second series is absolutely convergent on $\sigma = 1$, since it is merely a rearrangement of
\[ \sum_{n=1}^{x} \frac{1}{\log n}. \]
which is absolutely convergent by comparison with

$$\sum_{p \leq x} \frac{1}{p} \sim \frac{1}{x} \log x \sim \frac{1}{x} \log x$$

Hence also

$$\log x = \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \frac{1}{p} \mu(p) \frac{1}{p \log p} \sim \sum_{p \leq x} \frac{1}{p} \mu(p)$$

$$= \sum_{p \leq x} \frac{1}{p} \log \frac{1}{p} (\sigma = 1, t = 0).$$

Taking exponentials,

$$x = \exp \left( \sum_{p \leq x} \frac{1}{p} \right)$$

(3.15.1)

i.e. Euler's product holds on $a = 1$, except at $t = 0$.

As $s = 1$ the product is, of course, not convergent, but we can obtain an asymptotic formula for its partial products, viz.

$$\prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{s} \sim \frac{e^{\gamma}}{\log x}. \quad (3.15.2)$$

To prove this, we have to prove that

$$f(x) = - \log \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = \log \log x + o(1).$$

Now we have proved that

$$g(x) = \sum_{n \leq x} \frac{\Lambda(n)}{n} \sim - \log x + o(1).$$

Also

$$f(x) - g(x) = \sum_{p \leq x} \frac{1}{p} \mu(p) \sum_{p \leq x} \frac{1}{p \log p} \sim \sum_{p \leq x} \frac{1}{p} \mu(p) \sum_{p \leq x} \frac{1}{p} \frac{1}{p \log p}$$

$$\leq \sum_{p \leq x} \frac{1}{p} \mu(p) \frac{1}{p \log p}.$$
the strength of Theorem 3.10, giving a zero-free region around \( 1 + it \) solely in terms of an estimate for \( \zeta(s) \) in a neighbourhood of \( 1 + it \).

Ingham's method in §3.4 is of special interest because it avoids any reference to the behaviour of \( \zeta(s) \) near \( 1 + 2it \). It is possible to get quantitative zero-free regions in this way, by incorporating simple sieve estimates (Balasubramanian and Ramachandra [1]). Thus, for example, the analysis of §3.8 yields

\[
\sum_{\rho} \frac{\log p}{p^{2}} \left[ 1 + \cos(\gamma \log p) \right] \leq \frac{1}{\sigma} - \frac{1}{\sigma - \beta} + O(\log \gamma).
\]

However one can show that

\[
\sum_{\rho, \pi \leq X} \left[ 1 + \cos(\gamma \log p) \right] \gg \frac{X}{\log X}
\]

for \( X \gg \delta \), by using a lower bound of Chebyshev type for the number of primes \( X < p \leq 2X \), coupled with an upper bound \( O(h/\log h) \) for the number of primes in certain short intervals \( X < p \leq X + h \). One then derives the estimate

\[
\sum_{\rho, \pi \leq X} \frac{\log p}{p^{2}} \left[ 1 + \cos(\gamma \log p) \right] \geq \frac{\log n}{\sigma - 1},
\]

and an appropriate choice of \( \sigma = 1 + (A/\log \gamma) \) leads to the lower bound \( 1 - \beta = (\log \gamma)^{-1} \).

3.18. Another approach to zero-free regions via sieve methods has been given by Montgomery (11.3.1) (the sieve is symmetric, but not the advantage of applying to the wider regions discussed in §3.17, 6.15 and 6.19. One may also obtain zero-free regions from a result of Montgomery (1. Theorem 11.3) on the proliferation of zeros. Let \( \eta(t, w, \delta) \) denote the number of zeros \( \rho = \beta + it \) of \( \zeta(s) \) in the rectangle \( 1 - w < \beta < 1, \quad t - \delta < y < t + \delta \). Suppose \( \delta = 0 \) is any zero with \( \beta = \beta > 0 \), and that \( \delta \) satisfies \( 1 - \beta < \delta < (\log \gamma)^{-1} \). Then there is some \( \tau \) with \( \delta < \tau < 1 \) for which

\[
n(t, \tau, \delta) > n(2t, \tau, \delta) \geq \frac{t^{2}}{2(1 - \beta)}.
\]

Roughly speaking, this says that if \( \beta \) is small, there must be many other zeros near either \( 1 + iy \) or \( 1 + 2iy \). Montgomery gives a more precise version of this principle, as do Ramachandra [1] and Balasubramanian and Ramachandra [8]. To obtain a zero-free region

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one couples hypotheses of the type used in Theorem 3.10 with Jensen's Theorem, to obtain an upper bound for \( n(t, \tau, \delta) \). For example, the bound

\[
\zeta(s) \ll (1 + T^{-1 - \varepsilon}) \log T, \quad T = |t| + 2,
\]

which follows from Theorem 4.11, leads to

\[
n(t, \tau, \delta) \ll r \log T + \log \log T + \log \frac{1}{\delta}.
\]  

On choosing \( \delta = (\log \gamma)/(\log \gamma) \), a comparison of (3.18.1) and (3.18.2) produces Theorem 3.8 again.

One can also use the Epstein zeta-function of §2.18 and the Maass-Selberg formula (2.18.9) to prove the non-vanishing of \( \zeta(s) \) for \( \tau = 1 \). For, if \( s = \frac{1}{2} + it \) and \( \phi(s) = 2(2x) = 0 \), then

\[
|\psi(\frac{1}{2} + it)|^2 = \psi(1) = \psi(1 - 1) = |\psi(1 + 1)|^2 = 0,
\]

by the functional equation (2.19.1). Thus (2.19.9) yields

\[
\int_{\gamma} B(z, s) \cdot B(z, w) \frac{dx dy}{y^2} = 0
\]

for any \( w \neq s, 1 - s \). This, of course, may be extended to \( w = s \) or \( w = 1 - s \) by continuity. Taking \( w = \frac{1}{2} - it \), we obtain

\[
\int_{\gamma} B(z, s) \frac{dx dy}{y^2} = 0
\]

so that \( B(z, s) \) must be identically zero. This however is impossible since the Fourier coefficient for \( n = 1 \) is

\[
8\pi y \sum_{\rho} K_{\rho}(2x)/(\Gamma(s),
\]

according to (2.18.5), and this does not vanish identically. The above contradiction shows that \( (2s) = 0 \). One can get quantitative estimates by such methods, but only rather weak ones. It seems that the proof given here has its origins in unpublished work of Selberg.

3.19. Lemma 3.12 is a version of Perron's formula. It is sometimes useful to have a form of this in which the error is bounded as \( x \to \infty \).
Theorem of Hadamard

Lemma 3.19. Under the hypotheses of Lemma 3.12 one has
\[
\sum_{n \in \mathbb{N}} d_n = 1 \int_{-i/T}^{e^{i/T}} f(x + iw) \frac{e^{iw}}{w} dw + O\left(\frac{x}{T(x - 1)}\right) + O\left(\frac{x}{T(x - N)}\right).
\]

This follows at once from Lemma 3.12 unless \( x - N = O(x/T) \). In the latter case one merely estimates the contribution from the term \( n = N \) as
\[
\int_{-i/T}^{e^{i/T}} f(x + iw) \frac{e^{iw}}{w} dw = \int_{-i/T}^{e^{i/T}} \frac{e^{iw}}{N} \left(1 + O\left(\frac{|w|}{T}\right)\right) \frac{dw}{w} = \frac{e^{-i/T}}{N} \left(\log x + O(1)\right)
\]
and the result follows.

IV

Approximate Formulae

4.1. In this chapter we shall prove a number of approximate formulae for \( \zeta(s) \) and for various sums related to it. We shall begin by proving some general results on integrals and series of a certain type.

4.2. Lemma 4.2. Let \( F(x) \) be a real differentiable function such that \( F'(x) \) is monotonic, and \( F'(x) \geq m > 0 \), or \( F'(x) \leq -m < 0 \), throughout the interval \([a, b]\). Then
\[
\left|\int_a^b e^{itF(x)} dx\right| \leq \frac{4}{m}.
\]

Suppose, for example, that \( F'(x) \) is positive increasing. Then by the second mean-value theorem
\[
\int_a^b \cos[F(x)] dx = \frac{1}{F'(x)} \int_a^b F(x) \cos[F(x)] dx
\]
and the modulus of this does not exceed \( 2/m \). A similar argument applies to the imaginary part, and the result follows.

4.3. More generally, we have

Lemma 4.3. Let \( F(x) \) and \( G(x) \) be real functions, \( G(x)F'(x) \) monotonic, and \( F'(x)/G(x) \geq m > 0 \), or \( -m < 0 \). Then
\[
\left|\int_a^b G(x)e^{itF(x)} dx\right| \leq \frac{4}{m}.
\]
The proof is similar to that of the previous lemma.

The values of the constants in these lemmas are usually not of any importance.

4.4. Lemma 4.4. Let \( F(x) \) be a real function, twice differentiable, and let \( F''(x) \geq 0 > 0 \), or \( F''(x) \leq -c < 0 \), throughout the interval \([a, b]\). Then
\[
\left|\int_a^b e^{itF(x)} dx\right| \leq \frac{4}{c}.
\]
Consider, for example, the first alternative. Then \( F'(x) \) is steadily increasing, and so vanishes at most once in the interval \((a, b)\), say at \( c \). Let

\[
I = \int_a^b e^{\phi(x)} \, dx = \int_{c-\delta}^{c+\delta} + \int_{c-\delta}^{c-\delta} - I_1 + I_2 + I_3,
\]

where \( \delta \) is a positive number to be chosen later, and it is assumed that \( a + \delta \leq c \leq b - \delta \). In \( I_3 \)

\[
P'(x) = \int_c^x P''(t) \, dt \geq r(x-c) \geq rb.
\]

Hence, by Lemma 4.2,

\[
|I_3| \leq \frac{b}{\theta r^2}.
\]

\( I_1 \) satisfies the same inequality, and \( |I_2| \leq 25 \). Hence

\[
|I| \leq \frac{a}{\theta r^2} + 25.
\]

Taking \( \delta = 2r^{-\frac{1}{2}} \), we obtain the result. If \( c < a + \delta \), or \( c > b - \delta \), the argument is similar.

4.5. **Lemma 4.5.** Let \( F(c) \) satisfy the conditions of the previous lemma, and let \( Q(c) = F'(c) \) be monotonic, and \( |Q(c)| \leq M \). Then

\[
\left| \int_c^{c+\delta} e^{\phi(x)} Q(x) \, dx \right| \leq \frac{2M}{\sqrt{\theta}}.
\]

The proof is similar to the previous one, but uses Lemma 4.3 instead of Lemma 4.2.

4.6. **Lemma 4.6.** Let \( F(x) \) be real, with derivatives up to the third order. Let

\[
0 < \lambda_1 \leq F'(x) \leq \lambda_2,
\]

or

\[
0 < \lambda_1 \leq F'(x) \leq \lambda_2,
\]

and

\[
|F''(c)| \leq M,
\]

throughout the interval \((a, b)\). Let \( F'(c) = 0 \), where

\[
a \leq c \leq b.
\]

Then in the case (4.6.1)

\[
\int_c^{c+\delta} e^{\phi(x)} \, dx = (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2} F'(c)} + O\left( \frac{1}{F''(c)} \lambda_1 \right) + O\left( \frac{1}{F''(c)} \lambda_2 \right)
\]

(4.6.2) the factor \( e^{\phi(x)} \) is replaced by \( e^{\phi(c)} \). If \( F'(x) \) does not vanish on \([a, b]\) then (4.6.3) holds without the leading term.

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If \( F'(x) \) does not vanish on \([a, b]\) the result follows from Lemmas 4.2 and 4.4. Otherwise either (4.6.1) or (4.6.2) shows that \( F'(x) \) is monotonic, and so vanishes at only one point \( c \). We put

\[
\int_c^{c+\delta} e^{\phi(x)} \, dx = \int_c^{c+\delta} + \int_{c-\delta}^{c+\delta} - I_1 + I_2 + I_3,
\]

assuming that \( a + \delta \leq c \leq b - \delta \). By (4.2.1)

\[
\int_c^{c+\delta} = O\left( \frac{1}{F'(c+\delta)} \right) \quad (4.6.4)
\]

Similarly

\[
\int_c^{c-\delta} = O\left( \frac{1}{F'(c-\delta)} \right).
\]

Also

\[
\int_{c-\delta}^{c+\delta} = \int_{c-\delta}^{c+\delta} \exp\left( i(x-c)F'(c) + i(x-c)^{\frac{1}{2}} F''(c) \right) + O\left( \frac{1}{F''(c)} \right)
\]

\[
\int_{c-\delta}^{c+\delta} = \int_{c-\delta}^{c+\delta} \exp\left( i(x-c)F'(c) + O\left( \frac{1}{F''(c)} \right) \right)
\]

\[
\exp\left( i(x-c)F'(c) \right) dx = O\left( \frac{1}{F''(c)} \right).
\]

Supposing \( F''(c) \geq 0 \), and putting

\[
\int_{c-\delta}^{c+\delta} = O\left( \frac{1}{F''(c)} \right)
\]

the integral becomes

\[
\int_{c-\delta}^{c+\delta} = \int_{c-\delta}^{c+\delta} \frac{1}{F''(c)} \, dx
\]

\[
\int_{c-\delta}^{c+\delta} = \int_{c-\delta}^{c+\delta} \frac{1}{F''(c)} \, dx + O\left( \frac{1}{F''(c)} \right)
\]

\[
= \frac{2\pi}{F''(c)} + O\left( \frac{1}{F''(c)} \right).
\]

Taking \( \delta = \lambda_1 \delta_2 \), the result follows.

If \( b - \delta < c < b \), there is also an error

\[
\int_{c-\delta}^{c+\delta} = O\left( \frac{1}{F''(c)} \right) + O\left( \frac{1}{F''(c)} \right)
\]

and similarly if \( a < c < a + \delta \).
4.7. We now turn to the consideration of exponential sums, i.e., sums of the form
\[ \sum e^{fn(x)}, \]
where \( f(x) \) is a real function. If the numbers \( f(n) \) are the values taken by \( f(x) \) at \( x = n \), and let \( f'(x) = a, f''(x) = b \). Then
\[ \sum e^{fn(x)} = \sum \int_{a - \gamma x \leq b + \epsilon x} e^{fn(x)} e^{b \epsilon x} dx + O(\log(b + \epsilon + 2)). \tag{4.7.1} \]
where \( \gamma \) is any positive constant less than 1.

We may suppose without loss of generality that \( \gamma - 1 < a \leq \eta \), so that \( \nu \geq 0 \); for if \( k \) is the integer such that \( \gamma - 1 < a - k \leq \eta \), and
\[ h(x) = f(x) - kx, \]
then (4.7.1) is
\[ \sum e^{hn(x)} = \sum e^{h(n)x} e^{b \epsilon x} dx + O(\log(b + \epsilon + 2)), \]
where \( \rho' = a - k \), \( \beta' = b - k \), i.e., the same formula for \( h(x) \).

In (2.1), let \( h(x) = e^{bx} \), Then
\[ \sum e^{bn(x)} = \sum e^{bn(x)} e^{b \epsilon x} dx + O(\log(b + \epsilon + 2)). \]
Also
\[ a - 1 \leq a - k = \frac{1}{\pi} \sum \frac{\sin \beta x}{x} \]
if \( x \) is not an integer; and the series is bounded uniformly, so that we may multiply by an integrable function and integrate term-by-term. Hence the second term on the right is equal to
\[ -2 \pi \sum \frac{\sin \beta x}{x} e^{b \epsilon x} (f(x)) dx \]
which may be written
\[ -2 \pi \sum \frac{\sin \beta x}{x} e^{b \epsilon x} (f(x)) dx = \sum \frac{1}{\pi} \int_{a - \gamma x \leq b + \epsilon x} e^{b \epsilon x} e^{b \epsilon x} (f(x)) dx. \]
The integral may be written
\[ \frac{1}{2\pi} \int \frac{f'(x)}{f(x) - \nu} dx - \frac{1}{2\pi} \int \frac{f'(x)}{f(x) + \nu} dx + \frac{1}{2\pi} \int \frac{f'(x)}{f(x) - \nu} dx. \]
\[ \text{van der Corput (1).} \]

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Since \( f'(x) \) is steadily decreasing, the second term is
\[ O\left( \frac{\beta}{\beta + 1} \right), \]
by applying the second mean-value theorem to the real and imaginary parts. Hence this term contributes
\[ O\left( \frac{1}{\nu} \right) + O\left( \frac{\beta}{\nu} \right) + O\left( \frac{\beta}{\nu^2} \right). \]
Similarly the first term is \( O(\beta(\nu - \beta)) \) for \( \nu \gg 1 \), and this contributes
\[ O\left( \frac{1}{\nu} \right) + O\left( \frac{\beta}{\nu} \right) + O\left( \frac{\beta}{\nu^2} \right). \]
Finally
\[ \sum \frac{1}{\nu} \int e^{\beta x/(\nu - x)} f'(x) dx \]
and the integrated terms are \( O(\log(\beta + 2)) \). The result therefore follows.

4.8. As a particular case, we have

Lemmas 4.5. Let \( f(x) \) be a real differentiable function in the interval \( [a, b] \), let \( f'(x) \) be monotonic, and let \( f'(a) < \theta < f'(b) \). Then
\[ \sum e^{bn(x)} = \int e^{b \epsilon x} dx + O(1). \tag{4.8.1} \]
Taking \( \gamma < 1 - \theta \), the sum on the right of (4.7.1) either reduces to the single term \( \nu = 0 \), or, if \( f''(x) \geq 0 \) or \( \leq 0 \) throughout \( [a, b] \), it is null, and
\[ \int e^{b \epsilon x} dx + O(1) \]
by Lemma 4.2.

4.9. Theorem 4.6. Let \( f(x) \) be a real function with derivatives up to the third order. Let \( f'(x) \) be steadily decreasing in \( a < x < b \), and \( f'(b) = \beta \). Let \( x \) be defined by
\[ f'(x) = \nu \quad (a < x < b). \]
\[ \text{van der Corput (2).} \]
Theorem 4.11. We have
\[ \mathcal{I}(\nu) = \sum_{\nu = 1}^{\infty} \frac{1}{\nu^s} + O(\nu^{-s}) \quad (4.11.1) \]
uniformly for \( s > \sigma > 0, |t| < 2\pi\nu/C, \) when \( C \) is a given constant greater than 1.

We have, by (3.5.3),
\[ \mathcal{I}(\nu) = \sum_{\nu = 1}^{\infty} \frac{1}{\nu^s} + O(\nu^{-s}) \quad (4.11.2) \]

The sum
\[ \sum_{\nu = 1}^{\infty} \frac{1}{\nu^s} = \sum_{\nu = 1}^{N^{-1}} \frac{N^{-s}}{\nu^s} + O(\nu^{-s}) + O(\nu^{-s}) \]
is of the form considered in the above lemma, with \( g(\nu) = -\nu \), and
\[ f(\nu) = -\log \nu \quad (4.11.3) \]

Thus
\[ \mathcal{I}(\nu) = \sum_{\nu = 1}^{\infty} \frac{1}{\nu^s} + O(\nu^{-s}) \quad (4.11.4) \]

Hence
\[ \sum_{\nu \leq N} \frac{1}{\nu^s} = \mathcal{I}(\nu^s) + O(\nu^{-s}) \quad (4.11.5) \]

Hence
\[ \zeta(s) = \sum_{\nu \leq N} \frac{1}{\nu^s} + O(\nu^{-s}) + O(s^{-1}) \quad (4.11.6) \]

Making \( N \to \infty \), the result follows.

† Hardy and Littlewood (3).
4.12. For many purposes the sum involved in Theorem 4.11 contains too many terms (at least $A(t)$) to be of use. We therefore consider the result of taking smaller values of $a$ in the above formulae. The form of the result is given by Theorem 4.9, with an extra factor $g(n)$ in the sum. If we ignore error terms for the moment, this gives

$$
\sum_{n \leq x} g(n)e^{i\theta(n)} \sim e^{it} \sum_{n \geq x} \left( \frac{\sin(\pi n \tau)}{\pi n} \right) \tilde{g}(n).
$$

Taking

$$
g(n) = n^{-\sigma}, \quad f(n) = \frac{\log n}{2\pi},
$$

$$
f'(n) = \frac{t}{2\pi}, \quad f''(n) = -\frac{t^2}{2\pi^2},
$$

and replacing $a$, $b$ by $N$, and $i$ by $-i$, we obtain

$$
\sum_{n \geq x} \frac{1}{n^\sigma} \sim \int_{\sigma+i\mathcal{R}} \frac{e^{2\pi i t}}{t^\sigma} \frac{1}{\Gamma(1)} \sum_{n \geq x} \frac{1}{n^\sigma}.
$$

Now the functional equation is

$$
\zeta(s) = \chi(s)\zeta(1-s),
$$

where

$$
\chi(s) = 2^{s-1}\pi^{-s/2} \sec \frac{\pi s}{2} \Gamma(s).
$$

In any fixed strip $0 < \sigma < \beta$, as $t \to \infty$

$$
\log \Gamma(\sigma + it) = (\sigma + it - \frac{1}{2}) \log(2\pi t) - \frac{1}{2} \log 2\pi + O\left(\frac{1}{t}\right).
$$

Hence

$$
\Gamma(\sigma + it) = e^{-(\sigma - 1/2) \pi t^2 / \log(2\pi) + O\left(\frac{1}{t}\right)},
$$

and

$$
\chi(s) = \left(\frac{2\pi}{t}\right)^{s/2} e^{-(\sigma - 1/2) \pi t^2} + O\left(\frac{1}{t}\right).
$$

The above relation is equivalent to

$$
\sum_{n \geq x} \frac{1}{n^\sigma} \sim \chi(s) \sum_{n \geq x} \frac{1}{n^\sigma}.
$$

The formulae therefore suggest that, with some summable error terms,

$$
\zeta(s) \sim \sum_{n \geq x} \frac{1}{n^\sigma} \chi(s) \sum_{n \geq x} \frac{1}{n^\sigma},
$$

where $2\pi x = |t|$.

4.13. Actually the result is that

$$
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq x} \frac{1}{n^s} + O(x^{-\epsilon} + O(x^{-\epsilon} \log k) + O(x^{-\epsilon} y^{-\epsilon})),
$$

for $0 < \epsilon < 1$. This is known as the approximate functional equation.†

4.13. THEOREM 4.13. If $\zeta$ is a positive constant,

$$
0 < \epsilon < 1, \quad 2\pi x = t, \quad x > h > 0, \quad y > k > 0,
$$

then

$$
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq x} \frac{1}{n^s} + O(x^{-\epsilon} \log k) + O(x^{-\epsilon} y^{-\epsilon}).
$$

This is an imperfect form of the approximate functional equation in which a factor $\log k$ appears in one of the $O$-terms; but for most purposes it is quite sufficient. The proof depends on the same principle as Theorem 4.9, but Theorem 4.9 would not give a sufficiently good $O$-result, and we have to recompute the integrals which occur in this problem. Let $t > 0$. By Lemma 4.10

$$
\sum_{n \leq x} \frac{1}{n^\sigma} \sim \sum_{n \leq x} \int_0^1 \frac{e^{2\pi i t \log n}}{n^\sigma} du + O\left(\frac{x^{-\epsilon} \log \left(\frac{x}{N} + 1\right)}{x}\right),
$$

and the last term is $O(x^{-\epsilon} \log t)$. If $2\pi x y > 1$, the first term is $v = 0$, i.e.

$$
\sum_{n \leq x} \frac{1}{n^\sigma} \sim \int_{\sigma+i\mathcal{R}} \frac{e^{2\pi i t \log n}}{n^\sigma} du + O\left(\frac{x^{-\epsilon} \log \left(\frac{x}{N} + 1\right)}{x}\right),
$$

Hence by (4.11.2)

$$
\zeta(s) = \sum_{n \leq x} \frac{1}{n^\sigma} + \sum_{n \geq x} \int_0^1 \frac{e^{2\pi i t \log n}}{n^\sigma} du + O(x^{-\epsilon} \log \log x) + O(N^{-\epsilon}),
$$

since

$$
x^{-\epsilon} \log(1-x) = O(x^{-\epsilon} \log t),
$$

Now

$$
\zeta(s) = \sum_{n \leq x} \frac{e^{2\pi i t \log n}}{n^\sigma} du + O(x^{-\epsilon} \log \log x) + O(N^{-\epsilon}),
$$

and by Lemma 4.3

$$
\int_S \frac{e^{2\pi i t \log n} \sin(\pi n \tau)}{n^\sigma} du = O\left(\frac{x^{-\epsilon}}{\log(2\pi x)}\right) = O\left(\frac{x^{-\epsilon}}{\log(2\pi x)}\right),
$$

$$
\int_S \frac{e^{2\pi i t \log n} \cos(\pi n \tau)}{n^\sigma} du = O\left(\frac{x^{-\epsilon}}{\log(2\pi x)}\right).
$$

† Hardy and Littlewood (1, 4, 6), Siegel (3).
Hence
\[
\sum_{1 \leq a \leq N - 1} \int_{\frac{1}{2}}^{1 - \frac{1}{a}} y^\alpha \frac{dy}{\sqrt{1 - y^2}} \, dx = \left( \frac{\pi}{2} \right)^{-1} \Gamma(1 - \alpha) \sum_{1 \leq a \leq N - 1} \left( \frac{1}{a^2} + \frac{1}{a^2 - 1} \right)
+ O(N^{-1} \log N) + O\left( \frac{\sqrt{\pi}}{N} \right) + O\left( \frac{1}{N} \sum_{1 \leq a \leq N - 1} \frac{1}{a^2 - 1} \right)
- \frac{1}{2} \pi \Gamma(1 - \alpha) \left( \sum_{1 \leq a \leq N - 1} \frac{1}{a^2 - 1} + O(N^{-1} \log N) + O\left( \frac{\sqrt{\pi}}{N} \right) \right).
\]
There is still a possible term corresponding to \( y - \eta < r \leq y + \eta \); for this, by Lemma 4.5,
\[
\int_{\frac{1}{2}}^{1 - \frac{1}{a}} x^{\alpha - 2 \eta^2 / 2} \, dx = O\left( \frac{\pi}{2} \right)^{-1} \Gamma(1 - \alpha) \left( \frac{\eta}{a^2} \right)^{1 - \alpha},
\]
giving a term
\[
O\left( \left( \frac{\eta}{a^2} \right)^{1 - \alpha} \right).
\]
Finally we can replace \( r \leq y - \eta \) by \( r \leq x \) with error
\[
O\left( \left( \frac{\eta}{a^2} \right)^{1 - \alpha} \right).
\]
Also for \( t > 0 \)
\[
x^\alpha \sin \left( \frac{1}{2} \pi t \right) = 2 \pi \sin \left( \frac{1}{2} \pi t \right) \Gamma(1 - \alpha)
= 2 \pi \sin \left( \frac{1}{2} \pi t \right) \Gamma(1 - \alpha) + O(\sin \left( \frac{1}{2} \pi t \right) \, \Gamma(1 - \alpha)).
\]
Hence the result follows on taking \( N \) large enough.

It is possible to prove the full result by a refinement of the above methods. We shall not give the details here, since the result will be obtained by another method, depending on contour integration.

4.14. Complex-variable methods. An extremely powerful method of obtaining approximate formulæ for \( I(\sigma) \) is to express \( I(\sigma) \) as a contour integral, and then move the contour into a position where it can be suitably dealt with. The following is a simple example.

\textit{Alternative proof of Theor. 4.11.} We may suppose without loss of generality that \( x \) is half an odd integer, since the last term in the sum, which might be affected by the restriction, is \( O(x^{-\frac{1}{2}}) \), and so is the possible variation in \( x^{-\frac{1}{2}} \).

Suppose first that \( \sigma > 1 \). Then a simple application of the theorem of residue shows that
\[
I(\sigma) = \sum_{\sigma < a \leq N} \int_{\sigma - \frac{1}{2}}^{\sigma + \frac{1}{2}} z^{-\sigma} \cot \pi z \, dz = - \frac{1}{2} \int_{\sigma}^{\sigma + \frac{1}{2}} z^{-\sigma} \cot \pi z \, dz
= - \frac{1}{2} \int_{\sigma}^{\sigma + \frac{1}{2}} \left( \pi z^{-\sigma + 1} \right) \cot \pi z \, dz = \frac{\pi}{2} \int_{\sigma}^{\sigma + \frac{1}{2}} z^{\sigma - 1} \, dz
\]
The final formula holds, by the theory of analytic continuation, for all values of \( z \), since the last two integrals are uniformly convergent in any finite region. In the second integral we put \( z = x + it \), so that
\[
|\cot \pi z + i| = \frac{2}{1 + x^2} < 2 e^{-\pi t},
\]
and
\[
|x| = |x| e^{-\pi t} < x^{-\pi} \frac{\log \pi x}{\log x} < x^{-\pi t/2}.
\]
Hence the modulus of this term does not exceed
\[
\int_{\sigma}^{\sigma + \frac{1}{2}} e^{\pi t} \, dt = \frac{2}{2 e - \pi}.
\]
A similar result holds for the other integral, and the theorem follows.

It is possible to prove the approximate functional equation by an extension of this argument; we may write
\[
\cot \pi z = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{i\pi n z}}{\pi n},
\]
and this is \( O(x^{-\frac{1}{2}}) \) if \( 2(k+1)x - |\pi| > A \), i.e. for comparatively small values of \( x \), if \( x \) is large. However, the rest of the argument suggested is not particularly simple, and we prefer another proof, which will be more useful for further developments.

4.15. Theorem 4.15. The approximate functional equation (4.12.4) holds for \( 0 < \alpha < 1, x > 0, y > 0 \). It is possible to extend the result to any strip \( -k < \alpha < k \) by slight changes in the argument.

\textit{For} \( \sigma > 1 \)
\[
I(\sigma) = \sum_{\sigma - \frac{1}{2} < \frac{1}{2}} \int_{\sigma + \frac{1}{2}}^{\sigma + \frac{1}{2}} (\frac{1}{2} \pi t)^{-\sigma} \, dt
= \frac{\pi}{2} \int_{\sigma}^{\sigma + \frac{1}{2}} z^{\sigma - 1} \, dz.
\]
we have
\[ \arctan \left( \frac{1 + x}{u} \right) + \frac{u}{\eta} > \arctan \left( \frac{1 + x}{c} \right) + c \]
\[ = \frac{x}{2} + \frac{c}{2} - \arctan \left( \frac{1 + x}{c} \right) = \frac{x}{2} + A, \]
since for \(0 < \theta < 1\)
\[ \arctan \theta < \frac{\theta}{\frac{1}{\theta} - 1} = \frac{\theta}{1 - \theta}. \]
Hence
\[ \int_{c_1} = O\left( \frac{\sqrt{\eta}}{\pi} \int e^{-u^2} \, du \right) + O\left( \frac{\sqrt{\eta}}{\pi} \int e^{-c^2} \, du \right) = O\left( \frac{c}{\sqrt{\pi}} \right). \]
Finally consider \( C_0\). Here \( w = i \eta + e^i \pi \), where \( \lambda \) is real, \( |\lambda| \leq \sqrt{2} \eta. \)
Hence
\[ w^{-1} = \exp \left\{ (s-1) \left[ i \eta + \log(\eta + e^i \pi) \right] \right\} \]
\[ = \exp \left\{ (s-1) \left[ \frac{\lambda}{2} + \frac{\lambda^2}{8} \eta - \frac{1}{8} \eta^2 + O(\frac{1}{\eta}) \right] \right\} = O\left( \frac{\sqrt{\eta}}{\pi} \right). \]
Also
\[ w^{-1} \rightarrow \eta^{-1} \frac{1}{s-1} (u \geq 0, \quad \eta \geq \frac{1}{\sqrt{2} \eta}) \quad \text{or} \quad w^{-1} \rightarrow e^{(2\pi \gamma \eta)^{-1}} \]
Hence the part with \( |u| > \frac{1}{2} \eta \) is
\[ O\left( \eta e^{-\lambda \eta} \int_{-\infty}^{\infty} \exp \left\{ \frac{1}{2} \lambda^2 \eta + O(\frac{1}{\eta}) \right\} \right) \]
\[ = O\left( \eta e^{-\lambda \eta} \int_{-\infty}^{\infty} e^{-\lambda \eta} \, d\eta \right) = O\left( \eta^2 e^{-\lambda \eta} \right). \]
The argument also applies to the part \( |u| > \frac{1}{2} \eta \) on this part. If not, suppose, for example, that the contour goes too near to the pole at \( w = 2 \pi i \). Take it round an arc of the circle \( |w - 2 \pi i| = \frac{1}{2} \eta \).

On this circle,
\[ w = 2 \pi i + \frac{1}{2} \eta \]
and
\[ \log(w - \log \psi) = -\frac{1}{\log \psi} \left( \log(\psi) + (\psi - \log(\psi)) \right) + O(1). \]
Since \( \frac{m\psi - t}{3\psi} = \frac{2m\psi - t}{3\psi} = O(1), \)
this is
\[ -\psi + (\psi - \log(\psi)) + O(1). \]
Hence
\[ \log(w - \log \psi) + O(\psi - \log \psi). \]
The contribution of this part is therefore
\[ O(\psi - \log \psi)^2. \]
Since
\[ e^{-(\log \psi)^2} (1 - s) = O(\psi - \log \psi)^2, \]
we have now proved that
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} + O(\psi - \log \psi)^2 + O(1). \]
The \( O \)-terms are
\[ O(\psi) + O(\log(1 + (\psi - \log \psi)^2)) \]
\[ = O(\psi + O(\psi)) + O(1) = O(\psi - \log \psi)^2. \]
This proves the theorem in the case considered.

To deduce the case \( x > y \), change \( s \) into \( 1 - s \) in the result already obtained.
Then
\[ \zeta(1 - s) = \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} + O(2n-3). \]
Multiplying by \( \chi(s) \), and using the functional equation and
\[ \chi(s) = \chi(1-s), \]
we obtain
\[ \zeta(s) = \chi(s) \sum_{n=1}^{\infty} \frac{1}{n^s} + \frac{1}{n^s} + O(\psi)^2. \]
Interchanging \( s \) and \( y \), this gives the theorem with \( x > y \).

4.16. Further approximations.† A closer examination of the above analysis, together with a knowledge of the formulae of § 2.10, shows that the \( O \)-terms in the approximate functional equation can be replaced by an asymptotic series, each term of which contains trigonometrical functions and powers of \( t \) only.

† Riemann (9).
where $\Gamma$ is a contour including the points 0 and $a$. Now

$$\log \phi(w) = (a-1) \log \left( 1 + \frac{w}{a} \right) + \frac{1}{2} \pi i (w - a).$$

Hence for $|w| < \frac{a}{2}$ we have

$$R(\log \phi(w)) < |a-1| \log \frac{a}{2} + \frac{1}{2} \pi |w| \leq \frac{1}{4} \pi |w|.$$

Let $|w| < \frac{a}{2}$, and let $\Gamma$ be a circle with centre $w = 0$, radius $\rho_{\phi}$, where

$$\rho_{\phi} = \frac{a}{2}.$$

Then

$$r_\phi(w) = O(1) \left( \frac{a}{2} \right)^{|w|}.$$

The function $e^{-\lambda \phi(w)}$ has the minimum $(2\pi/2N\pi)^{1/2}$ for $\rho = (2N\pi/6)^{1/2}$; $\rho_\phi$ can have this value if

$$\left( \frac{a}{2} \right)^{1/2} \lesssim \frac{2N\pi}{6} \lesssim \frac{3}{2} \pi.$$

Hence

$$r_\phi(w) = O\left( \left( \frac{a}{2} \right)^{|w|} \right), \quad \left\{ N \lesssim \frac{27}{2}, \quad |w| \lesssim \frac{20}{2N\pi} \left( \frac{a}{2} \right)^{1/2} \right\}.$$

For $|w| < \frac{a}{2}$ we can also take $\rho_\phi = \frac{a}{2}$; giving

$$r_\phi(w) = O\left( \left( \frac{a}{2} \right)^{|w|} \right) = O\left( \left( \frac{a}{2} \right)^{|w|} \right).$$

Consider the integral along $C_\phi$, and take $c = 2 - \frac{1}{2}$. Then

$$\int_{C_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{C_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{C_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{C_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} + \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w}.$$

If $|w| - 1 > A$ on $C_\phi$, the last integral is, as in the previous section,

$$\int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w} = \int_{\gamma_\phi} e^{-\lambda \phi(w)} \frac{dw}{w}.$$

for $N < A$. The case where the contour goes near a pole gives a similar result, as in the previous section.
This is all times the coefficient of \( F^* \) in
\[
\int_{\Theta} \frac{1}{x^2} (x^2 - 2 + 2x^2 + i) \exp \left( \sum_{n=1}^{N} \frac{1}{2} \frac{1}{(n^2 + 1)} \right) \frac{d\omega}{\omega^{2m+1}}.
\]

Let $\psi(x) = \frac{\cos(\sqrt{x^2 - a^2})}{\sqrt{x}}$

where
\[
\psi(c) = \frac{\cos(\sqrt{\frac{x^2}{a^2} - 1})}{\sqrt{x}}.
\]

Then we obtain
\[
r^{-\frac{1}{2}} (2\pi)^{n} 2 \pi \left( \sum_{n=1}^{N} \frac{1}{n!} \frac{1}{(n-2)!} \right) \sum_{\nu=0}^{N-1} \frac{1}{(n-2)!} \left( \frac{d}{dx} \psi^{(n)} \right)_{x=0} - \sum_{\nu=0}^{N-1} \frac{1}{(n-2)!} \left( \frac{d}{dx} \psi^{(n)} \right)_{x=0}
\]

Denoting the last sum by $S_n$, we have the following result.

**Theorem 4.16.** If $0 < \sigma < 1$, $m = \sqrt{\pi(2\sigma)}$, and $N < At$, where $A$ is a sufficiently small constant,
\[
\psi(\theta) = \psi(\theta - \pi) \sum_{n=0}^{N} \frac{1}{n!} \frac{1}{(n-2)!} + \sum_{\nu=0}^{N-1} \frac{1}{(n-2)!} \left( \frac{d}{dx} \psi^{(n)} \right)_{x=0} - S_n + O(\Theta^{-1}).
\]

**4.17. Special cases**

In the approximate functional equation, let $\sigma = \frac{1}{2}$ and $x = y = \theta(2m)^{1/2}$.

Then (4.12.4) gives
\[
\psi(\frac{1}{2} + i0) = \sum_{n=0}^{N} \frac{1}{n!} \frac{1}{(n-2)!} \frac{1}{(n+1)!} \left( \frac{d}{dx} \psi^{(n)} \right)_{x=0} + O(\Theta^{-1}).
\]
4.18. A different type of approximate formula has been obtained by
Maulenhold. Instead of using finite partial sums of the original
Dirichlet series, we can approximate \( \zeta(n) \) by sums of the form
\[
\sum_{n \leq x} \frac{\phi(n)}{n^s},
\]
where \( \phi(x) \) decreases from 1 to 0 as \( x \) increases from 0 to 1. This reduces
considerably the order of the error terms. The simplest result of this type is
\[
\zeta(s) = 2 \sum_{n \leq x} \frac{1}{n^{s/2}} + x(\phi) \sum_{n \leq x} \frac{1}{n^{s/4}}
- \chi(\phi) \sum_{n \leq x \leq 2x} \frac{1}{n^{s/2}} + O\left(\frac{1}{x^{s/2}} + \frac{1}{x^{s/4}}\right),
\]
valid for \( 2xy = [x], |t| \geq (x+1)^4 \), \(-2 < s < 2\).

There is also an approximate functional equation for \( \zeta(n) \). This is
\[
(\zeta(n))^2 = \sum_{d|n} \phi(d) \chi(d) \sum_{d|n} \phi(d) + O(n^{-1} \log n),
\]
where \( 0 < \sigma < 1 \), \( x = \zeta(2n^{-1}) \), \( x \gg h \gg 0, y \gg h > 0 \). The proofs of
this are rather elementary.

NOTES FOR CHAPTER 4

4.19. Lemmas 4.2 and 4.4 can be generalized by taking \( F \) to be \( k \) times
differentiable, and satisfying \( |F^{(k)}(x)| > \beta > 0 \) throughout \([a, b]\). By
writing induction, in the same way that Lemma 4.1 was derived from
Lemma 4.2, one finds that
\[
\int_a^b e^{ixF(x)} \, dx \leq \lambda_0 \lambda_1^{-1/4},
\]
The error term \( O(\lambda_0 \lambda_1^{-1}) \) in Lemma 4.6 may be replaced by
\( O(\lambda_1^{-1}) \), which is usually sharper in applications. To do this one
does the same \( \lambda = \lambda_1^{-1} \) in the proof. It then suffices to show that
\[
\int_a^b e^{ixF(x)} \, dx \leq \lambda_0 \lambda_1^{-1} - 1,
\]
if \( f \) has a continuous first derivative and satisfies \( f(x) \leq x^3 \delta^{-1}, \)
\footnote{Maulenhold (1).}
\footnote{Hardy and Littlewood (6), Titchmarsh (21).}

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\( f'^{(s)} \leq x^{3s-1} \). Here we have written \( \lambda = \frac{1}{2} F'(x) \) and
\[ f(x) = F(x + c) - F(x) - \frac{1}{2} x^2 F''(x). \]

If \( \delta \leq (\lambda \delta)^{-1} \) then (4.19.1) is immediate. Otherwise we have
\[
\int_a^b e^{ixF(x)} \, dx = \int_a^b e^{ixF(x)} \, dx + \int_a^b e^{ixF(x)} \, dx + \int_a^b e^{ixF(x)} \, dx,
\]
valid for \( 2xy = [x], |t| \geq (x+1)^4 \), \(-2 < s < 2\).

The second integral on the right is trivially \( O(\lambda \delta^{-1}) \), while the third,
for example, is, on integrating by parts,
\[
\int_a^b e^{ixF(x)} \, dx = \int_a^b e^{ixF(x)} \, dx - \int_a^b e^{ixF(x)} \, dx + \int_a^b e^{ixF(x)} \, dx,
\]
valid for \( 2xy = [x], |t| \geq (x+1)^4 \), \(-2 < s < 2\).

as required. Similarly the error term \( O((b-a) \lambda_1 \lambda_1^{-1}) \) in Theorem 4.9 may
be replaced by \( O((b-a) \lambda_1 \lambda_1^{-1}) \).

For further estimates along these lines see Vinogradov [28, 88-91] and
Heath-Brown [11]; Lemmas 6 and 10. These papers show that the
error term \( O((b-a) \lambda_1 \lambda_1^{-1}) \) can be dropped entirely, under suitable
conditions.

Lemmas 4.2 and 4.8 have the following corollary, which is sometimes
useful.

Lemma 4.20. Let \( f(x) \) be a real differentiable function on the interval
\([a, b]\), let \( f'(x) \) be monotonic, and let \( 0 < c < |f'(x)| < 9 < 1 \). Then
\[
\sum_{x < c + b} e^{ixF(x)} \leq \lambda_0 \lambda_1^{-1}.
\]
4.20. Weighted approximate functional equations related to those mentioned in §4.18 have been given by Lavrill [1] and Heath Brown [3, Lemma 1]. As a typical example one has

\[ \zeta(s) = \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{d(n)n^{-s}}{n!} + \frac{1}{s-1} \sum_{n=1}^\infty \frac{d(n)n^{-s}}{n!} \frac{\Gamma(s)}{\Gamma(s+1)} + O(x^{1-\epsilon} \log x \epsilon^{1-\epsilon}) \]  

(4.20.1)

uniformly for \( t > 1, \ |\eta| \leq \frac{1}{4}, \ xy = (\frac{1}{2}t^2)x, \ y > 1 \), for any fixed positive integer \( k \). Here

\[ w_k(s) = \frac{1}{2\pi i} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \zeta(t) - \frac{1}{\Gamma(t)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \quad (x > \max(0, -\epsilon)) \]

The advantage of (4.20.1) is the very small error term.

Although the weight \( w_k(s) \) is a little awkward, it is easy to see, by moving the line of integration to \( c = -\frac{1}{2} \), for example, that

\[ w_k(s) = \begin{cases} O(u^{-1}) & (u \geq 1), \\ 1 + O(u) + O \left( u^{1/2 \log 2} e^{-|t|} \right) & (0 < u \leq 1) \end{cases} \]

uniformly for \( 0 < \eta \leq 1, \ t > 1 \). More accurate estimates are however possible.

To prove (4.20.1) one writes

\[ \sum_{n=1}^\infty d(n)n^{-s} \left( \frac{1}{x} \right)^{1/2} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \zeta(t) - \frac{1}{\Gamma(t)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

(\( c > \max(0, -\epsilon) \)),

and moves the line of integration to \( R(s) = -d, \ d > \max(0, \epsilon) \), giving

\[ \frac{1}{2\pi i} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \zeta(t) - \frac{1}{\Gamma(t)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

\[ + \zeta(s)^k \Gamma(s) \zeta(s+1)^k \]

The residue term is easily seen to be \( O \left( x^{1-\epsilon} \log^2 (2+x) e^{-\epsilon(t)} \right) \). In the integral we substitute \( x = -w, \ t = (1/2\pi i)x^{-1} \), and we apply the functional equation (2.6.4). This yields

\[ \frac{1}{2\pi i} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \frac{1}{\Gamma(k)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

\[ = -\frac{1}{2\pi i} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \frac{1}{\Gamma(k)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

\[ = -\frac{1}{2\pi i} \int_{-\frac{1}{2} - \infty}^{1/2 + \infty} \left( \frac{1}{\Gamma(k)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

(4.20.2)

as required.

Another result of the same general nature is

\[ \int (\eta + it)^{2k} = \sum_{m=-\infty}^\infty d(m)d(n)n^{-i\eta i \eta -1+it} W_k(m) + O(e^{\epsilon t/2}) \]

for \( t > 1 \) and any fixed positive integer \( k \), where

\[ W_k(s) = \frac{1}{2\pi i} \int_{1-\epsilon}^{1+\epsilon} \left( \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

This type of formula has the advantage that the cross terms which would arise on multiplying (4.20.1) by the complex conjugate are absent.

By moving the line of integration to \( R(s) = -d, \ d > \max(0, \epsilon) \), one finds that

\[ W_k(s) = 2 \int \left( \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

\[ = \frac{1}{2\pi i} \int_{1-\epsilon}^{1+\epsilon} \left( \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \frac{d(n)n^{-t}}{n!} \right)^k \frac{x^t e^{ixt}}{x} \frac{dt}{\Gamma(t)} \]

(4.20.3)
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4.21. We may write the approximate functional equation (4.18.1) in the form
\[ \zeta(s) = S(s, x) + \chi(s) N(1 - s, x) + R(s, x). \]
The estimate \( R(s, x) \ll x^{1 - \varepsilon} \log t \) has been obtained by Jutila (see Ivic [3]: §4.22) to be best possible for
\[ t \ll \frac{\log x}{x^{1 - \varepsilon}}. \]
Outside this range however, one can do better. Thus Jutila (in work to appear) has proved that
\[ R(s, x) \ll t^{1 - \varepsilon} (\log t)^{2} \log \left( 1 + \frac{x}{t} \right) + t^{-1} x^{1 - \varepsilon} (\gamma + \log 0) \]
for \( 0 < \varepsilon < 1 \) and \( s > 1 \). The corresponding result for \( x < t \) may be deduced from this, via the functional equation. For the special case \( x = y = t/2x \) one may also improve on (4.18.1). Motobashi [2], [3], and in work in the course of publication, has established some very precise results in this direction. In particular he has shown that
\[ 2(1 - s) R \left( s, \frac{t}{2x} \right) = - \left( \frac{1}{\pi} \right)^\chi \Delta \left( \frac{t}{2x} \right) + O(t^{-1}), \]
where \( \Delta(x) \) is the remainder term in the Dirichlet divisor problem (see §12.1). Jutila, in work to appear, cited above, gives another proof of this. In fact, for the special case \( s = \frac{1}{2} \), the result was obtained 49 years earlier by Taylor [1].

V

THE ORDER OF \( \zeta(s) \) IN THE CRITICAL STRIP

5.1. The main object of this chapter is to discuss the order of \( \zeta(s) \) as \( t \to \infty \) in the "critical strip" \( 0 < s < 1 \). We begin with a general discussion of the order problem. It is clear from the original Dirichlet series (1.1.1) that \( \zeta(s) \) is bounded in any half-plane \( s \geq 1 + \varepsilon > 1 \); and we have proved in (2.12.2) that
\[ \zeta(1) = O(n^{0}) \quad (s \geq \frac{3}{2}). \]
For \( s < \frac{3}{2} \), corresponding results follow from the functional equation
\[ \zeta(s) = o(s) \zeta(1 - s) \]
In any fixed strip \( x < s < \beta, \) as \( t \to \infty \)
\[ \zeta(s) \sim \left( \frac{t}{2\pi} \right)^{1 - s} \]
by (4.12.3). Hence
\[ \zeta(s) = O(t^{1 - s}) \quad (s \leq -\delta < 0), \] (5.1.1)
and
\[ \zeta(s) = O(t^{1 + s}) \quad (s \geq -\delta). \]
Thus in any half-plane \( s \geq \alpha \),
\[ \zeta(s) = O(t^{\alpha}), \quad k = k(\alpha), \]
i.e. \( \zeta(s) \) is a function of finite order in the sense of the theory of Dirichlet series.†
For each \( \sigma \) we define a number \( \mu(\sigma) \) as the lower bound of numbers \( \varepsilon \) such that
\[ \zeta(s + \varepsilon) = O(t^{\varepsilon}). \]
It follows from the general theory of Dirichlet series† that, as a function of \( \sigma, \mu(\sigma) \) is continuous, non-increasing, and convex downwards in the sense that no arc of the curve \( y = \mu(\sigma) \) has any point above its chord; also \( \mu(s) \) is never negative.
Since \( \zeta(s) \) is bounded for \( s \geq 1 + \varepsilon \) \( (\delta > 0), \) it follows that
\[ \mu(s) = 0 \quad (s > 1), \] (5.1.2)
and then from the functional equation that
\[ \mu(s) = 1 - s \quad (s < 0). \] (5.1.3)
These equations also hold by continuity for \( s = 1 \) and \( \sigma = 0 \) respectively.†

† See Titchmarsh, Theory of Functions, §§6.4, 6.41.
‡ Ibid., §§5.55, 7.41.
The chord joining the points \((0, \frac{4}{3})\) and \((1, 0)\) on the curve \(y = \mu(u)\) is \(y = \frac{1}{u} - 4u\). It therefore follows from the convexity property that
\[
\mu(u) \leq \frac{1}{u} - 4u \quad (0 < \sigma < 1).
\] (5.1.4)

In particular, \(\mu(\frac{1}{2}) \leq \frac{1}{2}\), i.e.
\[
(*) + (\text{ii}) = O(\text{ii})
\] (5.1.5)

for every positive \(c\).

The exact value of \(\mu(u)\) is not known for any value of \(u\) between 0 and 1. It will be shown later that \(\mu(\frac{1}{2}) < \frac{1}{2}\), and the simplest possible hypothesis is that the graph of \(\mu(u)\) consists of two straight lines
\[
\mu(u) = \frac{1}{u} - \sigma \quad (0 < \sigma < \frac{1}{2}),
\] (5.1.6)

This is known as Lindelöf’s hypothesis. It is equivalent to the statement that
\[
(*) + (\text{ii}) = O(\text{ii})
\] (5.1.7)

for every positive \(c\).

The approximate functional equation gives a slight refinement on the above results. For example, taking \(\sigma = \frac{1}{2}, \; x = y = \sqrt{\frac{1}{2}e^{2\pi}}\) in (4.12.4), we obtain
\[
(*) + (\text{ii}) = \sum_{n \in \mathbb{Z}, |n| \neq 0} \frac{1}{n^1 + O(1)} \sum_{n \in \mathbb{Z}} \frac{1}{n^1 + O(1)} - O(n^{-1})
\]
\[
= O\left( \sum_{n \in \mathbb{Z}, |n| \neq 0} 1 \right) + O(\text{ii})
\]
\[
= O(1).
\] (5.1.8)

5.2. To improve upon this we have to show that a certain amount of cancelling occurs between the terms of such a sum. We have
\[
\sum_{n \in \mathbb{Z}, |n| \neq 0} n^{-1} = \sum_{n \in \mathbb{Z}, n \text{ real}} n^{-1} = \sum_{n \in \mathbb{Z}, n \text{ complex}} n^{-1}
\]
and we apply the familiar lemma of ‘partial summation’. Let
\[
b_1, b_2, \ldots, b_n \geq 0,
\]
and
\[
a_n = a_1 + a_2 + \cdots + a_n
\]
where the \(a_i\)’s are any real or complex numbers. Then if
\[
|b_n| \leq M \quad (n = 1, 2, \ldots),
\]
\[
|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq M a_n,
\] (5.2.1)

For
\[
a_1 b_1 + \cdots + a_n b_n = b_1 a_1 + b_2 a_2 + \cdots + b_n (a_n - a_{n-1})
\]
\[
= a_1 (b_1 - b_n) + a_2 (b_2 - b_n) + \cdots + a_n (b_n - b_n) + a_n b_n,
\]

5.3. In the critical strip

\[|a_1 b_1 \cdots b_n b_o| \leq M (a_1 b_1 + \cdots + b_n - a_n) = M a_n.\]

If \(0 < b_1 \leq b_2 \leq \cdots \leq b_n\), we obtain similarly
\[|a_1 b_1 + \cdots + a_n b_n| \leq 2M a_n.\]

If \(a_n = e^{-a_1 \pi i}, b_n = n^{-1}\), where \(a > 0\), it follows that
\[
\frac{1}{n^{-1}} n^{-1} = \mathcal{O}(a^{-1} \max_{x \in [1, 2]} \frac{1}{x} \cdot e^{-a_1 \pi i}).
\] (5.3.2)

This raises the general question of the order of sums of the form
\[
\Sigma = \sum_{n \in \mathbb{Z}} e^{\pm i \sigma n} f(n),
\] (5.3.3)

when \(f(n)\) is a real function of \(n\). In the above case,
\[
f(n) = \frac{-i \log n}{2 \pi}.
\]

The earliest method of dealing with such sums is that of Weyl,† largely developed by Hardy and Littlewood.‡ This is roughly as follows. We can reduce the problem of \(\Sigma\) to that of
\[
\sum_{n \in \mathbb{Z}, \pm 1} \omega(n) e^{\pm i \sigma n},
\]
where \(\omega(n)\) is a polynomial of sufficiently high degree, say of degree \(k\). Now
\[
| \Sigma | \leq \sum_{n \in \mathbb{Z}} | e^{\pm i \sigma n} \omega(n) |
\]
\[
\leq \sum_{n \in \mathbb{Z}} | \omega(n) | e^{-a_1 \pi i}
\]
with suitable limits for the sums, and \(\omega(n) = e^{\pm i \phi(n)}\) is of degree \(k-1\). By repeating the process we ultimately obtain a sum of the form
\[
\sum_{n \in \mathbb{Z}, \pm 1} \omega(n) e^{\pm i \sigma n},
\]
We can now actually carry out the summation. We obtain
\[
| \Sigma | = \frac{1}{2} \sqrt{\frac{1}{\sigma} + \frac{1}{\sigma}} \leq \frac{1}{2 \pi} \sqrt{\frac{1}{\sigma} + \frac{1}{\sigma}}
\] (5.3.5)

If \(|\cos \sigma n|\) is small compared with \(b - a\), this is a favourable result, and can be used to give a non-trivial result for the original sum \(X\).

An alternative method is due to van der Corput.¶ In this method we approximate to the sum \(\Sigma\) by the corresponding integral
\[
\int_{\sigma}^{\sigma + \frac{1}{2}} e^{\pm i x} d\sigma,
\]
† Weyl (1), (2), § Littlewood (2), Landau (3), ¶ van der Corput (1)–(2), van der Corput and Koksma (1), Titchmarsh (8–12).
and then estimate the integral by the principle of stationary phase, or some such method. Actually the original sum is usually not suitable for this procedure, and intermediate steps of the form (5.2.8) have to be used.

Still another method has been introduced by Vinogradov. This is in some ways very complicated; but it avoids the k-fold repetition used in the Weyl-Hardy-Littlewood method, which for large k is very 'unecnomical'. An account of this method will be given in the next chapter.

5.3. The Weyl-Hardy-Littlewood method. The relation of the general sum to the sum involving polynomials is as follows:

**Lemma 5.3.** Let k be a positive integer.

\[ b/a \leq |t|^{1 - k} \quad \text{for} \quad b/a \leq 1, \]

and

\[ \sum_{n=1}^{N} \exp \left( - \frac{in}{a} \left( \frac{m}{a} \frac{1}{2} \frac{1}{a} + \cdots + \frac{(-1)^{k-1} i m^{k-1}}{a^{k-1}} \right) \right) \leq M \quad (\mu \leq b/a). \]

Then

\[ \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right| < AM. \]

For

\[ \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right| = \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right| \]

\[ = \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right| \leq M \sum_{n=1}^{N} \left| e^{i\theta x + \xi n^k} \right| \]

\[ = M \sum_{n=1}^{N} \left| e^{i\theta x + \xi n^k} \right| \leq M \sum_{n=1}^{N} \left| e^{i\theta x + \xi n^k} \right| \]

\[ \leq M \sum_{n=1}^{N} \left| e^{i\theta x + \xi n^k} \right| \leq M \sum_{n=1}^{N} \left| e^{i\theta x + \xi n^k} \right| \]

\[ \leq 2M \left| e^{i\theta x + \xi n^k} \right| \leq 2M \left| e^{i\theta x + \xi n^k} \right| \]

\[ \leq 2M \left| e^{i\theta x + \xi n^k} \right| \leq 2M e^{\theta x}. \]

5.4. In the Critical Strip

The simplest case is that of \( \zeta(1+i) \), and we begin by working this out. We require the case \( k = 2 \) of the above lemma, and also the following

**Lemma.** Let

\[ S = \sum_{n=1}^{N} \exp \left( \frac{m}{a} \frac{1}{2} \frac{1}{a} + \cdots + \frac{(-1)^{k-1} i m^{k-1}}{a^{k-1}} \right). \]

Then

\[ |S|^2 \leq \mu + 2 \sum_{n=1}^{N} \min \left( \mu, \left| \cos \xi n^k \right| \right). \]

For

\[ |S|^2 = \sum_{n=1}^{N} \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right|^2 = \sum_{n=1}^{N} \left( \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right). \]

Putting \( m' = m - r \), this takes the form

\[ \sum_{n=1}^{N} \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right|^2 \leq \sum_{n=1}^{N} \left| \sum_{n=1}^{N} e^{i\theta x + \xi n^k} \right|^2, \]

where, corresponding to each value of \( r \), \( m' \) runs over at most \( \mu \) consecutive integers. Hence, by (5.2.6),

\[ |S|^2 \leq \sum_{n=1}^{N} \min \left( \mu, \left| \cos \xi n^k \right| \right). \]

5.5. Theorem 5.5. \( |\zeta(1+i)| = O(h \log h) \).

Let \( 2^h \leq a < 2h, b < 2h, \) and let

\[ \mu = \left[ \frac{a}{b} \right]. \]

Then

\[ \Sigma = \sum_{n=1}^{N} e^{i\theta x + \xi n^k} = \sum_{n=1}^{N} e^{i\theta x + \xi n^k} + \cdots + \sum_{n=1}^{N} e^{i\theta x + \xi n^k} = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_{\mu'}, \]

where

\[ N - \left| \frac{b-a}{\mu'} \right| = \Sigma_0 \left( \frac{b-a}{\mu'} \right) = O(h^2). \]

By § 5.3, \( \Sigma_0 = O(M), \) where \( M \) is the maximum of

\[ S_{\mu} = \sum_{n=1}^{N} \exp \left( - \frac{m}{a} \frac{1}{2} \frac{1}{a} + \cdots + \frac{(-1)^{k-1} i m^{k-1}}{a^{k-1}} \right) \]

for \( m' \leq \mu. \) By § 5.4 this is

\[ O \left( \left[ \frac{a}{b} \right] \min \left( \mu, \left| \cos \xi n^k \right| \right) \right). \]
\[
\Sigma = O(N+1) + O\left[ \sum_{n_{\ell}} \frac{1}{\log t} \sum_{\gamma_{\ell}} \min\left\{ \mu_{\ell}, \frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} \right\} \right]
\]

\[
= O(N+1) + O\left[ (N+1) \left( \sum_{n_{\ell}} \frac{1}{\log t} \sum_{\gamma_{\ell}} \min\left\{ \mu_{\ell}, \frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} \right\} \right) \right]
\]

where \( \vartheta_{\ell} = 2\pi(\gamma_{\ell}+\mu_{\ell}) \).

Now
\[
\frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} - \frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} = \frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} \left( \frac{1}{(\gamma_{\ell}+\mu_{\ell})} - \frac{1}{(\gamma_{\ell}+\mu_{\ell})} \right)
\]

which, as \( \nu \) varies, lies between constant multiples of \( \vartheta_{\ell}/\mu_{\ell} \), or, by (5.5.1), of \( \vartheta_{\ell}/\mu_{\ell} \). Hence for the values of \( \nu \) for which \( \vartheta_{\ell}/(\gamma_{\ell}+\mu_{\ell}) \) lies in a certain interval \([\nu_{1}, \nu_{2}+1] \), the least value but one of
\[
\left\lfloor \frac{\vartheta_{\ell}}{2\pi(\gamma_{\ell}+\mu_{\ell})} \right\rfloor
\]
is greater than \( \vartheta_{\ell}/(\gamma_{\ell}+\mu_{\ell}) \), the least but two is greater than \( 2\vartheta_{\ell}/(\gamma_{\ell}+\mu_{\ell}) \), the least but three is greater than \( 3\vartheta_{\ell}/(\gamma_{\ell}+\mu_{\ell}) \), and so on to \( O(N) = O(\bar{d}) \) terms. Hence these values of \( \nu \) contribute
\[
\vartheta_{\ell} - \frac{\vartheta_{\ell}}{\mu_{\ell}+1} - \vartheta_{\ell} + \frac{\vartheta_{\ell}}{\mu_{\ell}+\nu_{2}}
\]

The number of such intervals \([\nu_{1}, \nu_{2}+1] \) is
\[
O((N+1) \frac{\vartheta_{\ell}}{\mu_{\ell}+1} + 1)
\]

Hence the \( \nu \)-sum is
\[
O((N+1) \log t) + O\left( \frac{1}{\log t} \right)
\]

Hence
\[
\Sigma = O((N+1) \log t) + O\left( \frac{1}{\log t} \right)
\]

If \( \nu = O(\bar{d}) \), the second term can be omitted. Then by partial summation
\[
\sum_{-1/2 < \gamma_{\ell}+\mu_{\ell} < 1/2} \frac{1}{\gamma_{\ell}+\mu_{\ell}+1} = O(\bar{d} \log t) \quad (\bar{d} \ll 2n).
\]

By adding \( O(\log t) \) sums of the above form, we get
\[
\sum_{n_{\ell} \in \{\text{even}, \text{odd}\}} \frac{1}{n_{\ell}+1} = O(\bar{d} \log t).
\]

5.8 IN THE CRITICAL STRIP

Also
\[
\sum_{n_{\ell} \in \{\text{odd}\}} \frac{1}{n_{\ell}+1} = o\left( \sum_{n_{\ell} \in \{\text{even}\}} \frac{1}{n_{\ell}+1} \right) = O(\bar{d}).
\]

The result therefore follows from the approximate functional equation.

5.6. We now proceed to the general case. We require the following lemma.

**Lemma 5.6.** Let
\[
f(z) = \mathfrak{a}z^{k-1} + \ldots
\]

be a polynomial of degree \( k \) with real coefficients. Let
\[
S = \sum \mathfrak{a}z^{m}
\]

where \( m \) ranges over at most \( \mu \) consecutive integers. Let \( K = 2^{\bar{d}} \). Then for \( k > 2 \)
\[
|S|^2 \ll 2^{k\mu} \mathfrak{a}^{k-1} + 2^{k\mu} \mathfrak{a}^{k-1} \sum_{r} \min(\mu_{r}, \frac{1}{\vartheta_{r}}) \left( \min(\mathfrak{a}r, (k-1)r_{r-1}) \right)
\]

where each \( \nu \) varies from 1 to \( n_{\ell} \). For \( k = 1 \) the sum is replaced by the single term \( \min(\mu_{r}, \frac{1}{\vartheta_{r}}) \).

We have
\[
|S|^2 = \frac{1}{2\pi} \sum_{\gamma_{\ell}} \frac{1}{(\gamma_{\ell}+\mu_{\ell})} \left( \sum_{\gamma_{\ell}} \frac{1}{(\gamma_{\ell}+\mu_{\ell})} \right) (m = m_{\ell}-r_{\ell-1})
\]

and, for each \( r_{\ell} \), \( m \) ranges over at most \( \mu \) consecutive integers. Hence by H"older's inequality
\[
|S|^2 \ll \left( \sum_{r} \frac{1}{\vartheta_{r}} \right)^{2/k} \left( \sum_{r} \frac{1}{\vartheta_{r}} \right)^{1-k/k} |S_{r}|^{k}
\]

where the dash denotes that the term \( r_{\ell} = 0 \) is omitted. Hence
\[
|S|^2 \ll 2^{k\mu} \mathfrak{a}^{k-1} \left( \sum_{\gamma_{\ell}} \frac{1}{(\gamma_{\ell}+\mu_{\ell})} \right) \left( \sum_{\gamma_{\ell}} \frac{1}{(\gamma_{\ell}+\mu_{\ell})} \right) |S_{r}|^{k}
\]

If the shoots are true for \( k-1 \), then
\[
|S_{r}|^{k} \ll 2^{k\mu} \mathfrak{a}^{k-1} \sum_{r} \min(\mu_{r}, \frac{1}{\vartheta_{r}}) (k-1)r_{r-1}
\]
Hence
\[
\|e\|^k \leq 2^{k^2 - 2k^3 - 1} + 2^{k^2 - 2k^3 - 2k^4 - 3} \sum_{m \in \mathbb{N}} \min(\mu, \csc(\pi d/m), r_{1-m}, x_j)
\]
and the result for \(k\) follows. Since by \(\S\) 3.4 the result is true for \(k = 2\), it holds generally.

5.7. Lemma 5.7. For \(a < b \leq 2a, b \geq 2, K = 2, a = 0, l \geq l_b,
\[
\Sigma = \sum_{n \geq a} n^2 = O(a^{1+2\log(\log(a^2))} + O(2^{a^2 - 2a^3 \log^2(a^2)}),
\]
If \(a \leq 2026\), then
\[
\Sigma = O(a) = O(a^{1+2\log(\log(a^2))})
\]
as required. Otherwise, let
\[
\mu = [2a^2 - 2a^3 \log^2(a^2)]
\]
and write
\[
\Sigma = \sum_{n \geq a} n^2 = \sum_{n \geq a} n^2 \mu \cdot \sum_{n \geq a} n^2 \mu \cdot \sum_{n \geq a} n^2 \mu \cdot \cdots = \Sigma_m.
\]
Then \(\Sigma_m = O(M),\) where \(M\) is the maximum, for \(\mu \leq \mu_m\), of
\[
S_m = \sum_{n \geq a} \exp\left(-\frac{m}{a + \mu} \left(\frac{1}{2(a + \mu)^2} \cdots \cdots \cdots \right)^{m^k}ight).
\]
By Lemma 5.5
\[
S_m = O(a^{1+2k} + \sum_{n \geq a} \min\left(\mu, \csc(\frac{k-1}{2}(a + \mu)^2)\right))
\]
Hence
\[
\Sigma = O(N^{1+2\log(\log(a^2))} + O(a^{1+2\log(\log(a^2)))}
\]
\[
= O(N^{1+2\log(\log(a^2))} + O(a^{1+2\log(\log(a^2))}))
\]
by Hölder's inequality.

Now as \(\nu\) varies,
\[
\frac{(k-1)^a}{(k-1)^a + (k-1)^b} \quad \text{lies between constant multiples of} \quad \frac{(k-1)^a}{(k-1)^a + (k-1)^b} \quad \text{and} \quad \frac{(k-1)^a}{(k-1)^a + (k-1)^b}
\]
for \(b \geq 2\), \(a = 0\), \(l \geq l_b\).

5.8. Theorem 5.8. If \(l\) is a fixed integer greater than 2, and \(L = 2^{l-1}\), then
\[
\zeta(s) = O(\sigma^{1+2\log(\log(\sigma^2)))}
\]
\(\sigma = 1+1/L\).
\[
(5.8.1)
\]
The second term in Lemma 5.7 can be omitted if
\(a \leq L(1+2\log(\log(\sigma^2)))
\]
Taking \(k = l\) and applying the result \(O(\log(\sigma))\), times we obtain
\[
\sum_{n=0}^{\sigma^2} n^{-\sigma} \leq O(N^{-1+2\log(\log(\sigma^2)))}
\]
\(\sigma = 1+1/L\).
\[
(5.8.2)
\]
Similarly, for \(k < l\), we find
\[
\sum_{n=0}^{\sigma^2} n^{-\sigma} = O(N^{-1+2\log(\log(\sigma^2)))}
\]
\(\sigma = 1+1/L\).
\[
(5.8.3)
\]
for \( t^{x+k+\frac{3}{2}} \log -t < N \leq t^{x+k+1} \log^{-s} t \). The error term here is at most
\[
O(N^{1-\frac{s}{k}} \log^{1-s} N) \quad \text{with}
\]
\[
\beta \triangleq -\left(1 - \frac{1}{k+2} + \frac{1}{(k+1)k}\right).
\]
Thus \( \beta \leq 1/L \). When \( k = 1-L \) we have
\[
\beta = \left(1 - \frac{2}{L} \right) + \frac{1}{(1+1)L} = \frac{1}{(1+1)L}.
\]
and for \( 2 \leq k \leq 1-L \) we have
\[
\beta = \left(1 - \frac{2}{2k} \right) + \frac{1}{(k+1)k} = \frac{k-1}{(k+1)(k+2)} < \frac{1}{(1+1)L}.
\]
It therefore follows, on summing over \( k \), that (6.8.2) holds for \( N \leq t^{1-\frac{s}{k}} \). Hence, by partial summation, we have
\[
\sum_{n \geq m} \frac{n - 1}{2k} \log^{s-1} \frac{n}{2k} = O\left(\frac{1}{2k} \log^{s-1}(\log^{s-1}(n))\right),
\]
and the theorem follows from the approximate functional equation.

### 5.9. van der Corput's method.
In this method we approximate to sums by integrals as in Chapter IV.

**Theorem 5.9.** If \( f(t) \) is real and twice differentiable, and
\[
0 < \lambda < f'(a) \leq \lambda b \quad \text{or} \quad \lambda a < f'(c) \leq \lambda b
\]
throughout the interval \([a, b] \), and \( b > a + 1 \), then
\[
\sum_{c \leq n < d} e^{2\pi i f(n)} = O(b(a-b)\lambda^{2}) + O(\lambda^{3}).
\]
If \( \lambda \geq 1 \) the result is trivial, since the sum is \( O(b(a-b)) \). Otherwise Lemmas 4.7 and 4.4 give
\[
O(\beta - \alpha + 2)\lambda^{2} + O(\log(\beta - \alpha + 2)),
\]
where
\[
\beta - \alpha - f'(a) - f'(b) = O(b(a-b)\lambda^{2}).
\]
Since \( \log(\beta - \alpha + 2) = O(\beta - \alpha + 2) \), the result follows.

## 5.10. IN THE CRITICAL STRIP

**Lemma 5.10.** Let \( f(t) \) be a real function, \( a < t < b \), and \( q \) a positive integer not exceeding \( b-a \). Then
\[
\left| \sum_{c \leq n < d} e^{2\pi i f(n)} \right| \leq \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}}.
\]
For convenience in the proof, let \( d_{q,n} \) denote \( d \) if \( n \leq a \) or \( a > b \). Then
\[
\sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}}
\]
the inner sum vanishing if \( n \leq a-q \) or \( n > b+q \). Hence
\[
\sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}} = \frac{1}{q} \left[ \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right]^{\frac{1}{2}}.
\]
Since there are at most \( b-a+1 \) values of \( n \) for which the inner sum does not vanish, this does not exceed
\[
\left( \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right)^{\frac{1}{2}}.
\]
Now
\[
\left( \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right)^{\frac{1}{2}} = \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}}.
\]
Hence
\[
\sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}} \leq \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \left| \frac{1}{q} \right|^{rac{1}{2}}.
\]
In the last sum, \( f(n+1) - f(n) \equiv f(t+1) - f(t) \), for given values of \( t \), \( 1 < t < \frac{1}{2} - 1 \), \( f(t+1) \) is, namely \( \mu = 1 \), \( \mu = 1 \), up to \( \mu = 1 \), \( \mu = 1 \), with a consequent value of \( n \) in each case. Hence the modulus of this sum is equal to
\[
\left( \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right)^{\frac{1}{2}} \leq \left( \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right)^{\frac{1}{2}}.
\]
Hence
\[
\left| \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right| \leq \left( \sum_{q < \infty} \sum_{n \equiv \pm 1 \pmod{q}} e^{2\pi i f(n)} \right)^{\frac{1}{2}},
\]
and the result follows.
5.11. Theorem 5.11. Let \( f(z) \) be real and have continuous derivatives up to the third order, and let \( \lambda_k < f^{(k)}(z) < \lambda_k + \delta \), and \( b - a > 1 \). Then

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O([\lambda_k(b-a)]^1) + O([\delta - \lambda_k]^1).
\]

Let

\[
g(z) = f(x+y) - f(x).
\]

Then

\[
g^{(k)}(z) = f^{(k)}(x+y) - f^{(k)}(x) = f^{(k)}(y),
\]

where \( x < y < x+y \).

Consequently, by Theorem 5.9,

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O([\lambda_k(b-a)]^1) + O([\delta - \lambda_k]^1),
\]

or the same for \(-g^{(k)}(z)\). Hence by Theorem 5.9,

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O([\lambda_k(b-a)]^1) + O([\delta - \lambda_k]^1).
\]

Hence, by Lemma 5.11,

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O\left(\frac{b-a}{y^2}\right) + O\left(\frac{b-a}{y^2} \sum_{\alpha \in \mathbb{C}^0} \left(\frac{\lambda_k}{\alpha}\right)^1 + \frac{\lambda_k}{\alpha}\right)
\]

\[
- O\left(\frac{b-a}{y^2}\right) + O\left(b(a-b)y^2\right) + O\left(\frac{b-a}{y^2}\right).
\]

The first two terms are of the same order in \( \lambda_k \) if \( b = [\lambda_k^2] \) provided that \( \lambda_k \leq 1 \). This gives

\[
O(\lambda_k(b-a)) + O(\lambda_k b) + O(\lambda_k b).
\]

as stated. The theorem is easily trivial if \( \lambda_k > 1 \). The proof also requires that \( b < a \). If this is not satisfied, then \( b-a = O(\lambda_k b) \),

\[
b-a = O(\lambda_k b)^j
\]

and the result again follows.

5.12. Theorem 5.12.

\[\ell(x) = e^{\alpha(x-b)}\]

Taking \( f(z) = -(3\pi)^{-1} \log z \), we have

\[f^{(k)}(z) = -\frac{t}{n^2}\]

Hence if \( b \leq a \) the above theorem gives

\[
\sum_{\alpha \in \mathbb{C}^0} a^{\alpha(z-b)} = O\left(\frac{1}{y^2}\right) + O\left(\frac{1}{y^2}\right) + O(\alpha(b-a)) + O(\alpha(t-a)),
\]

and the second term can be omitted if \( a < t \). Then by partial summation

\[
\sum_{\alpha \in \mathbb{C}^0} \frac{1}{y^2} = O(\lambda_k).
\]

Also, by Theorem 5.9,

\[
\sum_{\alpha \in \mathbb{C}^0} a^{\alpha(z-b)} = O(\lambda_k) + O(\alpha(t-a)),
\]

and hence by partial summation

\[
\sum_{\alpha \in \mathbb{C}^0} \frac{1}{y^2} = O\left(\frac{1}{y^2}\right) + O\left(\frac{1}{y^2}\right)
\]

Hence (5.12.1) is also true if \( 1 < a < t \). Hence, applying (5.12.1) \( O(\log t) \) times, we obtain

\[
\sum_{\alpha \in \mathbb{C}^0} \frac{1}{y^2} = O(\log \log t),
\]

and the result follows.

5.13. Theorem 5.13. Let \( f(z) \) be real and have continuous derivatives up to the \( k \)-th order, where \( b \geq 1 \). Let \( \lambda_k \leq f^{(k)}(z) \leq \lambda_k + \delta \). Then

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O([\lambda_k(b-a)]^{1+k}b) + O(\delta(b-a)^{1+k}b),
\]

where the constants implied are independent of \( b \).

If \( \lambda_k \geq 1 \) the theorem is trivial, as before. Otherwise, suppose the theorem true for all integers up to \( k-1 \). Let

\[
g(z) = f(x+y) - f(x).
\]

Then

\[g^{(k)}(z) = f^{(k)}(x+y) - f^{(k)}(x) = O(\delta),
\]

where \( x < y < x+y \). Hence

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = O(\lambda_k(b-a))^{1+k}b + O(\delta(b-a)^{1+k}b).
\]

Hence the theorem with \( k-1 \) for \( k \) gives

\[
\sum_{\alpha \in \mathbb{C}^0} e^{\alpha(z-b)} = A_1(\lambda_k(b-a))^{1+k}b + A_2(\delta(b-a)^{1+k}b)
\]

(writing constants \( A_1, A_2 \) instead of the \( \alpha \)'s). Hence

\[
\sum_{\alpha \in \mathbb{C}^0} \frac{1}{y^2} = O\left(\frac{1}{y^2}\right) + O\left(\frac{1}{y^2}\right)
\]

since \( \sum_{\alpha \in \mathbb{C}^0} \frac{1}{y^2} < \int_{\alpha} \frac{1}{y^2} \, dx = \frac{q^2}{y^2} \leq 2q \cdot \lambda_k b \)
for $K \geq 4$. Hence, by Lemma 5.10,

\[
\sum_{n \in \mathbb{C}^{+}} \frac{1}{\lambda_n^{k-1} \lambda_n^{(b-a)q}} = A_1(b-a)^{q} + A_2(b-a)^{1-q}A_4(b-a)^{q}A_4(b-a)^{q} + \frac{2}{3}A_4(b-a)^{2q} + O\left(\frac{(b-a)^{q}}{\lambda_n^{k-1}}\right)
\]

\[
\leq A_4(b-a)^{q} + A_4(b-a)^{q}A_4(b-a)^{q} + \frac{2}{3}A_4(b-a)^{2q} + O\left(\frac{(b-a)^{q}}{\lambda_n^{k-1}}\right)
\]

To make the first two terms of the same order in $\lambda_n$, let

\[
q = \left\lfloor \frac{\lambda_n^{k-1}}{4} \right\rfloor + 1
\]

Then

\[
\lambda_n^{k-1} \leq q \leq 2\lambda_n^{k-1},
\]

\[
\lambda_n^{2k-2} \leq 2\lambda_n^{k-1},
\]

and we obtain

\[
\left| \sum_{n \in \mathbb{C}^{+}} \frac{1}{\lambda_n^{k-1}} \right| \leq A_4(b-a)^{q}A_4(b-a)^{q} + \frac{2}{3}A_4(b-a)^{2q} + O\left(\frac{(b-a)^{q}}{\lambda_n^{k-1}}\right),
\]

This gives the result for $\kappa$; the constants are the same for $\kappa$ as for $\kappa-1$ if

\[
A_1(b-a)^{q} \leq A_4(b-a)^{q} < A_3,
\]

which are satisfied if $A_1$ and $A_3$ are large enough.

We have assumed in the proof that $q = b-a$, which is true if $2\lambda_n^{k-1} \leq b-a$. Otherwise

\[
\left| \sum_{n \in \mathbb{C}^{+}} \frac{1}{\lambda_n^{k-1}} \right| \leq 2\lambda_n^{k-1} \leq 2\lambda_n^{k-1},
\]

and the result again holds.

\section*{5.14. Theorem 5.14}

If $l \geq 3$, $b = 2^{t-1}$, $\sigma = 1 - l/(2L - 2)$,

\[
f(z) = O\left(\log z \cdot \sigma \right).
\]

We apply the above theorem with

\[
f(a) = \frac{(l-1)!}{2^{2a} \pi}, \quad f(\infty(z)) = \frac{(l-1)!}{2^{2\infty(z)} \pi}.
\]

If $n < a < b$, then

\[
\frac{(l-1)!}{2^{2n} \pi} \leq \left| f(a) \right| \leq \left| f(\infty(z)) \right| \leq \frac{(l-1)!}{2^{2\infty(z)} \pi},
\]

and we may apply the theorem with

\[
\lambda_n \asymp \frac{(l-1)!}{2^{2n} \pi}, \quad h = 2^n.
\]
5.15. Comparison between the Hardy-Littlewood result and the van der Corput result. The Hardy-Littlewood method shows that the function \( \mu(s) \) satisfies

\[
\mu \left( 1 - \frac{1}{2^k} \right) \leq \frac{1}{(k+1)2^k},
\]

and the van der Corput method that

\[
\mu \left( 1 - \frac{1}{2^k} \right) \leq \frac{1}{2^k - 2}.
\]

(5.15.1)

For a given \( k \), determine \( l \) so that

\[
1 - \frac{1}{2^k} - \frac{1}{2^k - l} < \frac{1}{2^k - 2}.
\]

Then (5.15.2) and the convexity of \( \mu(s) \) give

\[
\mu \left( 1 - \frac{1}{2^k} \right) \leq \frac{1}{2^k - 2} \left( \frac{1}{2^k - l} \right)^2 + \frac{1}{2^k - 2} \frac{1}{l} - \frac{1}{2^k - 2} \frac{1}{2^k - 2},
\]

\[
= \frac{2^k - l}{2^k - 2} \left( \frac{1}{2^k - l} \right)^2 + \frac{1}{2^k - 2} \frac{1}{l} - \frac{1}{2^k - 2} \frac{1}{2^k - 2},
\]

if

\[
(k+1)(2^k - l) < (k+1)2^k.
\]

Similarly, if \( k+1 \leq l \), this is true if

\[
(k+1)(2^k - l) < (k+1)2^k + 2.
\]

i.e., if

\[
k+1 \leq l - 1.
\]

Now

\[
l - 1 < 2^k - 2 < l - 1.
\]

If \( l \geq 8 \). Hence the Hardy-Littlewood result follows from the van der Corput result if \( l \geq 8 \).

For \( 4 \leq l \leq 7 \) the relevant values of \( 1 - \sigma \) are

- \( H \), 1, 1, \( \frac{2^k}{l} \)
- \( v \), d. C., \( \frac{1}{l} \), \( \frac{2^k}{l} \), \( \frac{2^k}{l} \)

The values of \( k \) and \( l \) in these cases are 3, 4, 5 and 6, 7 respectively. Hence \( k \leq l - 2 \) in all cases.

5.16. Theorem 5.16.

\((1 + \epsilon) = O \left( \frac{\log t}{\log \log t} \right)\)

We have to apply the above results with \( k \) variable; in fact it will be seen from the analysis of § 5.15 and § 5.14 that the constants implied in the \( O \)'s are independent of \( k \). In particular, taking \( a = 1 \) in (5.14.4), we have

\[\sum_{a < t \leq 2a} \frac{1}{n_{1/2}} = O \left( \frac{\log n}{\log \log n} \right) \quad (a < b \leq 2a)\]

uniformly with respect to \( k \), subject to (5.14.3). If

\[\frac{1}{2^{k+2} + k} < a \leq \frac{1}{2^{k+1} - k}\]

it follows that

\[\sum_{a < t < b} \frac{1}{n_{1/2}} = O \left( \frac{\log n}{\log \log n} \right) \quad (a < b \leq 2a)\]

Writing\[
\sum_{\mu(n) = \pm 1} \frac{1}{n_{1/2}} = \sum_{1 < n < l} + \sum_{l < n \leq \infty},
\]

and applying the above result with \( k = 2, 3, \ldots \), or \( r \), we obtain, since there are \( O(\log r) \) terms,

\[\sum_{\mu(n) = \pm 1} \frac{1}{n_{1/2}} = O \left( \frac{\log n}{\log \log n} \right) \quad (a < b \leq 2a)\]

Let \( r = \log \log t \). Then

\[2E \leq \exp \left( \frac{\log t}{\log \log t} \right) \leq \exp (\log t) \leq \exp (\log t) > A \log t,\]

Hence the above sum is bounded. Also

\[\sum_{n_{1/2} \leq \infty} \frac{1}{n_{1/2}} = O \left( \frac{\log t}{\log \log t} \right) = O \left( \frac{\log t}{(\log \log t)^2} \right)\]

This proves the theorem. The same result can also be deduced from the Weyl-Hardy-Littlewood analysis.
5.17. Theorem 5.17. For \( t > A \)
\[
\zeta(t) = O(t^\omega), \quad \sigma \geq 1 - \frac{\omega \log \log t}{\log t},
\]
\[
\zeta(t) = O(t^{\omega}), \quad \sigma \geq 1 - A_{1} \frac{\log \log t}{\log t},
\]
(with some \( A_{1} \)), and
\[
\zeta(t) = O\left(\frac{\log t}{\log \log t}\right).
\]
We observe that (5.14.1) holds with a constant independent of \( t \), and also, by the Phragmén–Lindelöf theorem, uniformly for
\[
\sigma \geq 1 - \frac{1}{(2L-2)},
\]
Let \( t \) be given (sufficiently large), and let
\[
l = \left\lfloor \frac{\log t}{\log \log t} \right\rfloor.
\]
Then
\[
L \leq n \leq \frac{1}{2} \log \frac{t}{2 \log \log t},
\]
and similarly
\[
L \geq \frac{1}{4} \log \frac{t}{\log \log t}.
\]
Hence
\[
\frac{L}{2L-2} \geq \frac{1}{2} \frac{\log t}{\log \log t} - \frac{1}{2L-2} \geq \frac{1}{2} \frac{\log t}{\log \log t} - \frac{1}{2} \frac{\log \log t}{\log \log t}
\]
for \( t > A \) (large enough). Hence if
\[
\sigma \geq 1 - \frac{(\log \log t)^3}{\log t}
\]
then
\[
\sigma \geq 1 - \frac{1}{2L-2}.
\]
Hence (5.14.1) is applicable, and gives
\[
\zeta(t) = O\left(\frac{\log t}{\log \log t}\right),
\]
\[
\zeta(t) = O(t^{\omega}),
\]
This proves (5.17.1). The remaining results then follow from Theorems 3.10 and 3.11, taking (for \( t > A \))
\[
\delta(t) = \frac{(\log \log t)^3}{\log t}, \quad \phi(t) = \delta \log \log t.
\]

5.18. In this section we reconsider the problem of the order of \( \zeta(1+it) \). Small improvements on Theorem 5.12 have been obtained by various different methods. Results of the form
\[
\zeta(1+it) = O(t^{\omega} \log \log t)
\]
with
\[
\omega \approx 163 \quad 27 \quad 229 \quad 19 \quad 15
\]
were proved by Walfisz (1), Titchmarsh (9), Phillips (1), Titchmarsh (24), and Min (1) respectively. We shall give here the argument which leads to the index \( \omega \). The main idea of the proof is that we combine Theorem 3.13 with Theorem 4.9, which enables us to transform a given exponential sum into another, which may be easier to deal with.

Theorem 5.18.
\[
\zeta(1+it) = O(t^{\omega} \log \log t) \quad \text{for large } t.
\]

Consider the sum
\[
\Sigma_{n} = \sum_{\gamma \leq \log \log t} \sum_{\gamma \leq \log \log t} \sum_{\gamma \leq \log \log t} \sum_{\gamma \leq \log \log t} e^{-\gamma \log n},
\]
where \( a \leq b \leq 2a, a < A \). By (5.10)
\[
\Sigma_{n} = \frac{\lambda}{|\gamma|} \left| \sum_{\gamma} \left( \sum_{n \leq \gamma} \frac{\lambda}{\gamma} \right) \right|,
\]
where \( g \leq \log \lambda \), and
\[
\Sigma_{n} = \sum_{a < \log \log t} e^{-\gamma \log \log t}.
\]

We now apply Theorem 4.9. to \( \Sigma_{n} \). We have
\[
f(x) = -\frac{1}{2 \pi} \log(x+r)-\log x, \quad f'(x) = -\frac{1}{2 \pi} \log(x+r)-\log x,
\]
\[
f''(x) = \frac{2 \pi}{2 \pi} - \frac{2 \pi}{2 \pi} = 0.
\]
We can therefore apply Theorem 4.9 with \( \lambda_{1} = t \pi, \lambda_{2} = t \pi \). Thus
\[
\Sigma_{n} = e^{-\lambda_{1} \pi} \sum_{n \leq \gamma} \frac{e^{-\lambda_{1} \pi}}{\gamma} + \lambda_{1} \sum_{n \leq \gamma} \frac{\lambda_{1} \pi}{\gamma} + \lambda_{1} \sum_{n \leq \gamma} \frac{\lambda_{1} \pi}{\gamma}.
\]
We see that the \( \log \log \log \) terms can be omitted, since it is \( O(\lambda_{1} \log \log t) \).

Consider next the sum
\[
\sum_{a < \log \log t} e^{-\gamma \log \log t} (a < \gamma < b).
\]

Note that the proof of the lemma in Titchmarsh (24) is incorrect. The lemma should be replaced by the corresponding theorem in Titchmarsh (16).
The numbers of the form \( (a, b) \) are given by

\[
\begin{align*}
\text{Theorem } & A.1. \quad \text{For } k = 0, 1, 2, \ldots, \text{ the number of } (a, b, c) \text{ such that } c = \sum_{i=0}^{k-1} (a_i + b_i) \\
\text{is given by} & \quad \sum_{a_0 = 0}^{k-1} \left( \sum_{b_0 = 0}^{k-a_0-1} \left( \sum_{a_1 = 0}^{k-a_0-b_0-1} \ldots \left( \sum_{a_{k-1} = 0}^{k-a_{k-2}-1} 1 \right) \ldots \right) \ldots \right) = \sum_{a_0 = 0}^{k-1} \binom{k-a_0}{k}.
\end{align*}
\]

\[\text{Hence,}\]

\[
\Phi(k) = \sum_{a_0 = 0}^{k-1} \binom{k-a_0}{k}.
\]

6.1 Theorem. Two more complicated arguments have been given.

6.1.1. \textbf{Theorem for Chapter 6}
By using Lemma 5.10 one may prove that
\[
A(p, q) = \left( \frac{p - 2p + q + 1}{2p + 2} \right)
\]
is an exponent pair whenever \((p, q)\) is, Similarly from Theorem 4.9, as sharpened in §4.19, one may show that
\[
B(p, q) = (q - 1, p + 1)
\]
is an exponent pair whenever \((p, q)\) is, providing that \(p + 2q > \frac{3}{2}\). Thus one may build up a range of pairs by repeated applications of these \(A\) and \(B\) processes. In doing this one should note that \(B(p, q) = (p, q)\).

Examples of exponent pairs are:
\[
\begin{align*}
B(0, 1) &= (1, 1), & A^2B(0, 1) &= (3, 3), \\
A^2B(0, 1) &= (3, 3), & A^3B(0, 1) &= (7, 7), & A^4B(0, 1) &= (13, 13), \\
A^4B(0, 1) &= (13, 13), & A^5B(0, 1) &= (25, 25), & A^6B(0, 1) &= (49, 49), \\
A^2A^2B(0, 1) &= (3, 3), & A^2A^2B(0, 1) &= (5, 5), & A^2A^2B(0, 1) &= (7, 7), \\
A^2A^2B(0, 1) &= (7, 7), & A^2A^2B(0, 1) &= (9, 9). & A^2A^2B(0, 1) &= (11, 11).
\end{align*}
\]

To estimate the sum \(\Sigma\) of §5.18 we may take
\[
f(x) = \frac{t}{2e} \log x, \quad y = \frac{t}{2e}, \quad s = 1,
\]
so that (5.20.1) holds for any \(c > 0\). The exponent pair \((44, 52)\) then yields
\[
\sum \leq t^4e^{\frac{t}{2e}}
\]
whence
\[
\sum_{s \leq c} n^{-s/2} = \mathcal{H}(n) = \mathcal{H}(n)
\]
uniformly for \((N, 1, f, y) \in \mathcal{F}(s, c)\).

We may observe that \(\lambda^N\) is the order of magnitude of \(f(x)\). It is immediate that \((0, 1)\) is an exponent pair. Moreover Theorems 5.9, 5.11, and 5.13 correspond to the exponent pairs \((1, 1), (q, 1), (q, 1),\) and
\[
\left( \frac{1}{2^2 - 2^2 - 2} \right)
\]
respectively.

Finally Bombieri, in unpublished work, has used a method related to that of §6.3, together with the bound
\[
\sum_{s \leq c} \exp \left[ 2\pi i (x + \beta z^2) \right] \leq P \log P,
\]
to prove (5.19.1). Secondly, (5.19.1) follows from the mean-value bound (7.24.4) of Iwaniec [1]. (This deep result is described in §7.24.) Heath-Brown [9] has shown that the weaker estimate \(\left| \mu_1 \right| \leq P\) follows from an argument analogous to Burgess's [1] treatment of character sums. Moreover the bound \(\mu_1 \leq \frac{1}{2}\), which is weaker still, but non the less non-trivial, follows from Heath-Brown's [4] fourth-power moment (7.21.1), based on Weil's estimate for the Kloosterman sum. Thus there are some extremely diverse arguments leading to non-trivial bounds for \(\mu_1\).
THE ORDER OF \( \psi(n) \) Chap. V

certain problems than any which can be got in this way, as we shall see in §§6.17–18. These unfortunately do not seem to help in the estimation of \( \psi(n) \).

6.21. The list of bounds for \( \psi(n) \) may be extended as follows.

\[
\begin{align*}
\psi(n) & = 0.161079 \ldots \quad \text{Wallis (1)}, \\
\psi(n) & = 0.164974 \ldots \quad \text{Titchmarsh (9)}, \\
\psi(n) & = 0.166451 \ldots \quad \text{Phillips (1)}, \\
\psi(n) & = 0.166450 \ldots \quad \text{Rankin (1)}, \\
\psi(n) & = 0.167329 \ldots \quad \text{Titchmarsh (24)}, \\
\psi(n) & = 0.167303 \ldots \quad \text{Moe (1)}, \\
\psi(n) & = 0.167301 \ldots \quad \text{Hafner (1)}, \\
\psi(n) & = 0.167316 \ldots \quad \text{Kolesnik (2)}, \\
\psi(n) & = 0.167327 \ldots \quad \text{Kolesnik (4)}, \\
\psi(n) & = 0.167300 \ldots \quad \text{Kolesnik (5)}.
\end{align*}
\]

The values \( \psi(n) \) were obtained by Chen (1), independently of Hafner, but a little later.

The estimates from Titchmarsh (24) onwards depend on bounds for multiple sums. In proving Lemma 5.10 the sum over \( r \) on the left of (5.10.1) is estimated trivially. However, there is scope for further savings by considering the sum over \( r \) as a two-dimensional sum, and using two-dimensional analogues of the \( A \) and \( B \) processes given by Lemma 2.10 and a theorem of Vinogradov. Indeed since further variables are introduced each time an \( A \) process is used, higher-dimensional sums will occur. Srinivasan (1) has given a treatment of double sums, but it is not clear whether it is sufficiently flexible to give, for example, new exponent pairs for one-dimensional sums.

VI

VINOGRAVODV’S METHOD

6.1. Still another method of dealing with exponential sums is due to Vinogradov. This has passed through a number of different forms of which the one given here is the most successful. In the theory of the \( \chi \)-function, the method gives new results in the neighbourhood of the line \( s = 1 \).

Let

\[
J(n) = a_1u_1^k + \ldots + a_nu_n^k
\]

be a polynomial of degree \( k \geq 2 \) with real coefficients, and let \( a \) and \( q \) be integers,

\[
S(q) = \sum_{\alpha < |q|} e(q\alpha),
\]

\[
J(q, s) = \int \frac{1}{q} \left| S(q) \right|^2 \, dq_1 \ldots dq_k.
\]

The question of the order of \( J(q, s) \) as a function of \( q \) is important in the method.

Since \( S(q) = O(q) \) we have trivially \( J(q, s) = O(q^s) \). Less trivially, we could argue as follows. We have

\[
(S(q))^2 = \sum_{m = 0}^\infty \sum_{n = 0}^\infty e^{nq\alpha} e^{m\beta},
\]

\[
|S(q)|^2 = \sum_{m_{1}, \ldots, m_{k}} e^{nq\alpha} e^{m_{1}q_{1} + \ldots + m_{k}q_{k}}.
\]

On integrating over the \( k \)-dimensional unit cube, we obtain a zero factor if any of the numbers

\[
m_1 + \ldots + m_k - a - \ldots - a_k \quad (k = 1, \ldots, k)
\]

is different from zero. Hence \( J(q, k) \) is equal to the number of solutions of the system of equations

\[
m_1 + \ldots + m_k = n_1 + \ldots + n_k \quad (k = 1, \ldots, k),
\]

where \( a < m_n \leq a+q, \quad a < n_n \leq a+q \).

But it follows from these equations that the numbers \( n_n \) are equal (in some order) to the numbers \( m_n \). Hence only the \( m_n \) can be chosen.

\[\text{Vinogradov (1926), Titchmarsh (1928), Hasse (1930).}\]
arbitrarily, and so the total number of solutions is \( O(q^h) \). Hence
\[
J(q, k) = O(q^h)\]
and
\[
J(q, l) = O(q^{2h-\lambda} J(q, k)) = O(q^{2h-\lambda}).
\]
This, however, is not sufficient for the application (see Lemma 6.8).

For any integer \( l \), \( J(q, l) \) is equal to the number of solutions of the equations
\[
m_1 \cdots m_l = a \cdots a^k (k = 1, 2, \ldots, b),
\]
where \( a < m_a \leq a + q, a < n_a \leq a + q \). Actually \( J(q, l) \) is independent of \( a \); for putting \( M_a = m_a - a, N_a = n_a - a \), we obtain
\[
\sum_{h=1}^{l} (M_a + a)^h = \sum_{h=1}^{l} (N_a + a)^h (k = 1, \ldots, b),
\]
which is equivalent to
\[
\sum_{h=1}^{l} M_a^h = \sum_{h=1}^{l} N_a^h (k = 1, \ldots, b),
\]
and \( 0 < M_a \leq q, 0 < N_a \leq q \). Clearly \( J(q, l) \) is a non-decreasing function of \( l \).

6.2. Lemma 6.2. Let \( m_1, \ldots, m_k, n_1, \ldots, n_k \) be two sets of integers, let
\[
s_h = \frac{1}{\pi} \sum_{h=1}^{l} a^h, \quad s_h' = \frac{1}{\pi} \sum_{h=1}^{l} a_h^h,
\]
and let \( a, a' \) be the \( h \)-th elementary symmetric functions of \( m \) and \( m' \), respectively. If \( |m_1| \leq q, |n_1| \leq q \), and
\[
|a_1 - a'_1| \leq q^{h-1} (k = 1, \ldots, b),
\]
then
\[
|a_1 - a'_1| \leq \frac{1}{2} (2by)^{h-1} (k = 1, \ldots, b).
\]

6.3. Lemma 6.3. Let \( 1 < G < c \), and let \( a_1, \ldots, a_h \) be integers satisfying
\[
1 < a_1 < a_2 < \cdots < a_h < G, \quad a_1 - a_h > 1.
\]
For each value of \( a \) \( 1 \leq a \leq k \) let \( m_a \) be an integer lying in the interval
\[
-a + (a-1)/G \leq m_a \leq -a + q/\theta,
\]
where \( 0 \leq a \leq q \). Then the number of sets of such integers \( m_1, \ldots, m_k \) for which the values of \( a \), \( k = 1, \ldots, b \), lie in given intervals of \( \log \phi \) exceeding \( q^{h-1} \), is \( \leq (4k)(q^{h-1}) \).

If \( x \) is any integer such that \( |x| > q \), the above lemma gives
\[
|x_1 - m_1| \cdots |x_k - m_k| \geq |x_1 - n_1| \cdots |x_k - n_k| \leq \frac{1}{2} \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]

6.4. Vinogradov's Method

Hence
\[
|a_1 - a'_1| \leq \frac{1}{2} \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]
\[
= \frac{1}{2} \sum_{h=1}^{l} 1 \leq \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]

6.5. Vinogradov's Method

Since \( (2k)^{-1} (l - 2) \leq \frac{1}{2} (2k)^{-1} \leq \frac{1}{2} (2l)^{-1} \)

since \( 2k/((2k-1)!) \leq 2 \) and \( j \geq 1 \). This proves the lemma.

6.3. Lemma 6.3. Let \( 1 < G < c \), and let \( a_1, \ldots, a_h \) be integers satisfying
\[
1 < a_1 < a_2 < \cdots < a_h < G, \quad a_1 - a_h > 1.
\]
For each value of \( a \) \( 1 \leq a \leq k \) let \( m_a \) be an integer lying in the interval
\[
-a + (a-1)/G \leq m_a \leq -a + q/\theta,
\]
where \( 0 \leq a \leq q \). Then the number of sets of such integers \( m_1, \ldots, m_k \) for which the values of \( a \), \( k = 1, \ldots, b \), lie in given intervals of \( \log \phi \) exceeding \( q^{h-1} \), is \( \leq (4k)(q^{h-1}) \).

If \( x \) is any integer such that \( |x| > q \), the above lemma gives
\[
|x_1 - m_1| \cdots |x_k - m_k| \geq |x_1 - n_1| \cdots |x_k - n_k| \leq \frac{1}{2} \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]

6.4. Vinogradov's Method

Hence
\[
|a_1 - a'_1| \leq \frac{1}{2} \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]
\[
= \frac{1}{2} \sum_{h=1}^{l} 1 \leq \sum_{h=1}^{l} (a_h - a'_h) |x|^h
\]

6.5. Vinogradov's Method

Since \( (2k)^{-1} (l - 2) \leq \frac{1}{2} (2k)^{-1} \leq \frac{1}{2} (2l)^{-1} \)

since \( 2k/((2k-1)!) \leq 2 \) and \( j \geq 1 \). This proves the lemma.
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Next, for a given value of $m_0$, the numbers $m_1, \ldots, m_{k-1}$ satisfy similar conditions with $k-1$ instead of $k$, and hence the number of values of $m_{k-1}$ is at most $(4kG)^{k-1} < (4kG)^{k-2}$. Proceeding in this way, we find that the total number of sets does not exceed

$$\sum_{m_{k-1}} (4kG)^{k-3}.$$ 

6.4. LEMMA 6.4. Under the same conditions as in Lemma 6.3, the number of sets of integers $m_1, \ldots, m_k$ for which the numbers $n_k$ (for $k = 1, \ldots, k$) lie in lengths of intervals not exceeding $c\log q$, where $c > 1$, does not exceed

$$2c(4kG)^{k-3}.$$ 

We divide the $n$th interval into

$$1 + \left[ \frac{2c(4kG)^{k-3}}{2c(4kG)^{k-3}} \right] \leq 2c(4kG)^{k-3}$$

parts, and apply Lemma 6.3. Since

$$\sum_{m_{k-1}} (2c(4kG)^{k-3}) = (2c)^{k-1}(4kG)^{k-3}$$

we have at most $(2c)^{k-1}(4kG)^{k-3}$ sets of sub-intervals, each satisfying the conditions of Lemma 6.3. For each set there are at most $(4kG)^{k-3}$ solutions, so that the result follows.

6.5. LEMMA 6.5. Let $k < l$, let $f(n)$ be as in 6.1, and let

$$I = \prod_{m_{k-l}} \left( Z_{m_{k-l}} \cdots Z_{m_{k-1}} \right) \left[(g^{[k-(l+1)]})^{d_1} \cdots d_{k-2} \right]$$

where

$$Z_{m_{k-l}} = \frac{\lambda(g)}{\lambda(g)},$$

and the $g$ satisfies (6.3.1) with $1 < G < q$. Then

$$I \leq 2k^{c\log q} \sum_{m_0, n_0} \left( g^{[k-(l+1)]} \cdots d_{k-2} \right).$$

We have

$$I \left( \sum_{m_0, n_0} \right) \leq 2k^{c\log q} \sum_{m_0, n_0} \left( g^{[k-(l+1)]} \cdots d_{k-2} \right) \left( g^{[k-l+1]} \right).$$

where $\sum_{m_0, n_0}$ is the number of solutions of the equations

$$m_0^2 + \cdots + m_k^2 - m_{k-l+1}^2 = N_k \quad (k = 1, \ldots, l).$$

for $m_0$ and $n_0$ in the interval $(g, 1)$.

Moreover, $N_k$ runs over those integers for which one can solve

$$N_k = m_0^2 + \cdots + m_k^2 - m_{k-l+1}^2.$$

where $m_0^2$ and $m_k$ lie in an interval $(a, a + q^{1-k_0})$. As in 6.1 we can shift each range through $a$, i.e. replace $a$ by $b$. Then $N_k$ ranges over at most $2k^{l-1}q^{1-l}$ values. Hence by Lemma 6.4, for given values of $n_{k-1}, \ldots, n_0$, the number of sets of $(m_1, \ldots, m_k)$ does not exceed

$$2k^{l-k}(q^{1-l})^2.$$ 

Also $(n_1, \ldots, n_k)$ takes not more than $(1 + 2^{-3k})^k \leq (2^{-1}q)^k$ values. Hence

$$\sum_{N_k} \sum_{N_0} \left( g^{[k-(l+1)]} \cdots d_{k-2} \right)$$

and the result follows.

6.6. LEMMA 6.6. The result of Lemma 6.5 holds whether the $g$ satisfies (6.3.1) or not, if $m$ has the value

$$M = \left[ \frac{\log q}{k \log 2} \right].$$

Since

$$|Z_{m_{k-l}}| \leq 2^{k-2}q^2 + 1 \leq 2^{k-2},$$

and

$$|Z_{m_{k-1}}| \leq 2^{k-2}q^2 + 1 \leq 2^{k-2},$$

it is sufficient to prove that

$$2^{k-2}q^2 \leq 2^{k-2}q^2 + 1 \leq 2^{k-2}q^2 + 1,$$

or that

$$2^{k-2}q^2 \leq 2^{k-2}q^2 + 1 \leq 2^{k-2}q^2 + 1,$$

or that

$$k \log q \leq kM \log 2 + 2k \left( k - 1 \right) \log 4,$$

or that

$$k \log q \leq kM \log 2 + k \left( k - 1 \right) \log 4,$$

Since

$$M \geq \frac{\log q}{k \log 2},$$

this is true if

$$k \log 2 \leq \frac{2k \left( k - 1 \right)}{k \log 2},$$

or

$$k \log 2 \leq \frac{2k \left( k - 1 \right)}{k \log 2},$$

which is true for $k \geq 2$.

6.7. LEMMA 6.7. The set of integers $(g_1, \ldots, g_k)$, where $k < l$, and each $g_i$ ranges over $(1, G)$, is said to be well-spaced if there are at least $k$ of them, say $g_{i_1}, \ldots, g_{i_k}$, satisfying

$$g_{i_k} - g_{i_{k-1}} > \frac{1}{2} (q^{1-l}).$$

The number of sets which are not well-spaced is at most $4k^{2-k}$.

Let $g_{i_1}, \ldots, g_{i_k}$ denote $g_{i_1}, \ldots, g_{i_k}$ arranged in increasing order, and let $f_i = g_i - g_{i-1}$. If the set is not well-spaced, there are at most $k - 2$ of the numbers $f_i$ for which $f_i > 1$. 

Consider those sets in which exactly \( k \) (with \( 0 \leq k \leq \varepsilon - 2 \)) of the numbers \( f_i \) are greater than 1. The number of ways in which these \( f_i \)'s can be chosen from the total \( \varepsilon - 1 \) is \( \binom{\varepsilon - 1}{k} \). Also each of the \( \varepsilon - k \) 's can take at most \( M \) values, and each of the rest at most \( 2 \) values. Since \( g_i \) takes at most \( M \) values, the total number of sets of \( g_i \)'s arising in this way is at most
\[
\binom{\varepsilon - 1}{k} M^{\varepsilon - k - 1} n^k.
\]
The total number of not well-spaced sets \( g_i \)'s is therefore
\[
\leq \sum_{k=0}^{\varepsilon - 2} \binom{\varepsilon - 1}{k} M^{\varepsilon - k - 1} n^k \leq \sum_{k=0}^{\varepsilon - 2} \frac{(\varepsilon - 1)^{k+1}}{k+1} M^{\varepsilon - k - 1} n^k
\]
\[
< \frac{1}{M} (1 + M^{-1}) < MN^{-1}.
\]
Since the number of sets of \( g_i \), corresponding to each set \( g_i \) is at most \( n \), the result follows.

6.8. Lemma 6.8. If \( l \geq \frac{4k^2}{\log 2} \) and \( M \) is defined by (6.6.1), then
\[
J(q, l) \lessdot \max\{1, M\} \log^{O(1)} \frac{y^{\log y}}{y^{\log 2} n^{\log 2}} \sum_{y^{\log 2} n^{\log 2}}^{y^{\log y}} \sum_{y^{\log 2} n^{\log 2}}^{y^{\log y}} \]
Suppose first that \( M \) is not less than 2, i.e. that \( q \geq 2^{30} \). Let \( \mu \) be a positive integer not greater than \( M - 1 \). Then
\[
\mu \lessdot \frac{\log q}{\log 2} - 1 . \quad \text{i.e.} \quad 2^{\mu} \lessdot q^{10}.
\]
Let
\[
S(p) = \sum_{p=\alpha}^{2^{\mu} n^{\log 2} \sum_{g=\alpha}^{q^{10}} |Z_{\mu}|}
\]
say. Then
\[
|S(p)| = \sum_{p=\alpha}^{2^{\mu} n^{\log 2} \sum_{g=\alpha}^{q^{10}} |Z_{\mu}|}
\]
where each \( g \) runs from 1 to \( 2^{\mu} \), and the sum contains \( 2^{\mu} \) terms.
We denote those products \( Z_{\mu} \in Z_{\mu} \) with well-spaced \( g \)'s by \( Z_{\mu} \).
The number of such \( Z_{\mu} \), say, does not exceed \( 2^{2\mu} \). In the remaining terms we divide each factor into two parts, so that we obtain products of the type \( Z_{\mu} \in Z_{\mu} \) each \( g \) lying in (1, \( 2^{\mu} \)). The number of such terms, \( Z_{\mu} \in Z_{\mu} \), say, does not exceed \( 2^{2\mu} \). By Lemma 6.7, the terms of this type with well-spaced \( g \)'s we denote by \( Z_{\mu} \), and the rest we subdivide again. We proceed in this way until finally \( Z_{\mu} \) denotes all the products of order \( M \), whether containing \( g \)'s well-spaced or not. We then have
\[
|S(p)| = \frac{2^{2\mu} n^{\log 2} \sum_{g=\alpha}^{q^{10}} |Z_{\mu}|}
\]
\[
|S(p)|^{1/2} \lessdot M \sum_{\mu=\alpha}^{M} n^{\log 2} \sum_{g=\alpha}^{q^{10}} |Z_{\mu}| \lessdot M \sum_{\mu=\alpha}^{M} n^{\log 2} \sum_{g=\alpha}^{q^{10}} |Z_{\mu}|. \quad (6.8.1)
\]
since we can start with an integer $m$ such that $2^m < E$. (Indeed, we may take $m = 1$.) Hence

$$J(q, l) < 2^{\theta + q^2} \cdot 2^{(\epsilon + 1)q^2},$$

and since

$$2^{\theta + q^2} \cdot 2^{(\epsilon + 1)q^2} < 2^{\theta + q^2},$$

the result follows.

If $M < 2$, i.e. $q < 2^m$, divide $S(q)$ into four parts, each of the form $S(q')$, where $q' < \frac{m}{4} < q^{10^{-5}}$. By Hölder's inequality,

$$|S(q)|^\theta \leq 4\sum |S(q')|^\theta \leq 4\sum |S(q')|^{\theta \cdot 10^{-5}} \sum |S(q')|^{\theta \cdot 10^{-5}}.$$

Integrating over the unit hypercube,

$$J(q, l) \leq 4^{\theta \cdot 10^{-5}} \sum |S(q')| \sum |S(q')|^{10^{-5}} \sum |S(q')|^{10^{-5}}$$

and the result again follows.

6.9. Lemma 6.9. If $r$ is any non-negative integer, and $l \geq \frac{1}{2} k^2 + \frac{1}{2} k + kr$, then

$$J(q, l) \leq K \log q \cdot q^{r \cdot 10^{-5}} \cdot k,$$

where

$$K = 4^{\theta \cdot 10^{-5}} \sum |S(q')|^{10^{-5}}.$$

This is obvious if $r = 0$, since then $S_0 = \frac{1}{2} k(k+1)$ and $J(q, l) \leq q^k$. Assuming that it is true up to $r-1$, Lemma 6.8 (in which $M \leq \log q$) gives

$$J(q, l) \leq K \log q \cdot q^{r \cdot 10^{-5}} \cdot k^2 \cdot k^{10^{-5}} \cdot k^{10^{-5}} \times q^{10^{-5}} \sum |S(q')|^{10^{-5}} \cdot k^{10^{-5}} \cdot k^{10^{-5}},$$

and the index of $q$ reduces to $2l - k(k+1) \cdot 5 \delta$. 

6.10. Lemma 6.10. If $l = \frac{1}{2} k^2 \log(k+1) + \frac{1}{2} k + k - 1, k \geq 7,$

$$J(q, l) \leq q^{(2 + \log \log k + \log k)} \cdot \log \log k \cdot \log \log \log k,$$

We have $\delta \leq \frac{1}{4}$ if

$$\log^2 (k+1) \leq \frac{k}{4},$$

i.e. if

$$\log (k+1) \leq \frac{k}{4}. $$

This true if

$$\log (k+1) \leq \frac{k}{4},$$

or if

$$r = \lfloor \log \log (k+1) \rfloor + 1, $$

Since

$$r = \lfloor \log \log (k+1) \rfloor + 1 \leq \log \log k \leq \log \log \log k,$$

and

$$\log K \leq K \log (4k + 4) \log l + \frac{1}{2} k \log (k+1) \log k \leq q \log l + 1 \log k \leq 10 \log k,$$

the result follows.

6.11. Lemma 6.11. Let $M$ and $N$ be integers, $N > 1$, and let $\phi(n)$ be a real function of $n$, defined for $M \leq n \leq M+N-1$, such that

$$\delta < \phi(n+1) - \phi(n) \leq \epsilon \alpha \quad (M \leq n \leq M+N-2),$$

where $\delta > 0$, $\alpha > 1$, $\epsilon \alpha < \frac{1}{2}$. Let $W > 0$. Let $\varepsilon$ denote the difference between $x$ and the nearest integer. Then the number of values of $n$ for which $\phi(n) \leq W$ is less than

$$\left(\delta \varepsilon \right)^{-1} (2W+1).$$

Let $a$ be a given real number, and let $b$ be the number of values of $n$ for which

$$a + \delta < \phi(n) \leq a + \alpha \delta,$$

for some integer $k$. To each $k$ corresponds at most one $n$, so that $G \leq a + \alpha \delta + 1$, where $a$ and $a + \delta$ are the least and greatest values of $n$. But clearly

$$\phi(M) \leq a + \alpha \delta,$$

whence

$$\phi(M) - a - \alpha \delta \leq \phi(M+N-1) - \phi(M) \leq (N-1) \delta,$$

and

$$G \leq (N-1) \delta + 1 \leq \delta W + 1.$$
by Hölder's inequality, where \( f \) is any positive integer.

We now write \( A_r = A_r(n) = f^{(r)}(n)/r! \) for \( 1 \leq r \leq k \), and define the \( k \)-dimensional region \( \Omega_k \) by the inequalities

\[ |\xi_r - A_r(n)| < \frac{\epsilon}{k^r} \quad (r = 1, \ldots, k). \]

If we set

\[ \delta(n) = f(n+1) - f(n) - (A_1 n^* + \ldots + A_k n^{k-1}), \]

then, by partial summation, we will have

\[ T(n) = \int_0^1 (S(q) \delta(q))e^{2\pi inq}dq. \]

We may then use this theorem together with the bound (6.12.1) to obtain

\[ \delta(q) = f'(p+1) - \sum_{r=1}^k \frac{p r^{r-1}}{k^r} \sum_{s=1}^1 \frac{p s^{r-1}}{k^s} \sum_{t=1}^1 \frac{p t^{r-1}}{k^t} S_r(p) \delta(q) e^{2\pi inq}. \]

where \( 0 < \epsilon < 1 \). If (6.12.6) holds it follows that

\[ |\delta(q)| < \epsilon \beta \equiv \epsilon \beta \sum \frac{p r^{r-1}}{k^r} \sum_{s=1}^1 \frac{p s^{r-1}}{k^s} \sum_{t=1}^1 \frac{p t^{r-1}}{k^t} S_r(p) \delta(q) e^{2\pi inq}, \]

by our choice of \( \epsilon \). We therefore have

\[ |\delta(q)| \leq \epsilon \sum \frac{p r^{r-1}}{k^r} \sum_{s=1}^1 \frac{p s^{r-1}}{k^s} \sum_{t=1}^1 \frac{p t^{r-1}}{k^t} S_r(p) \delta(q) e^{2\pi inq}. \]

Integrating over the region \( \Omega_k \), and dividing by its volume, we obtain

\[ |T(n)|^2 \leq (2k^{1/2} + 1) \sum_{\xi} \left| \int_{\Omega_k} (S(q) \delta(q))e^{2\pi inq}dq \right|^2. \]

The integral of \( |S(q)|^2 \) over \( \Omega_k \) is equal to its integral taken over the region obtained by subtracting any integer from each coordinate. We say that such a region is congruent (mod 1) to \( \Omega_k \). Now let \( n', n'' \) be two integers in the interval \( [P+1, P+Q-1] \), and let \( \Omega_k \) be the corresponding regions defined by (6.12.5). A necessary condition that \( \Omega_k \) should intersect with any region congruent (mod 1) to \( \Omega_k \) is that

\[ \frac{\lambda(n')}{\lambda(n'')} \leq \epsilon \lambda(n') \leq \epsilon \lambda(n''). \]

Let \( \phi(n) = A_1(n) - A_2(n) \). Then

\[ \phi(n+1) - \phi(n) = \frac{1}{k} \left( f^{(n+1)}(n) - f^{(a)}(n) \right) = \frac{f(a+1)}{k}, \]

where \( 0 < \epsilon \leq n+1 \). The conditions of Lemma 6.11 are therefore satisfied, with \( c = 2 \) and \( \beta = \lambda(k+1) \). Taking \( W = \phi(k+1) \), we see that the number of numbers \( n \) in \( [P+1, P+Q-1] \) for which (6.12.7) is satisfied, does not exceed

\[ 2k(k+1) \frac{2q}{P+Q-1} \leq 2k + 2 \frac{2q}{P+Q-1} \leq 3kq. \]

Since this is independent of \( n \), it follows that

\[ \int_0^1 \sum_{r=1}^k \frac{p r^{r-1}}{k^r} \sum_{s=1}^1 \frac{p s^{r-1}}{k^s} \sum_{t=1}^1 \frac{p t^{r-1}}{k^t} S_r(p) \delta(q) e^{2\pi inq}dq \]

\[ \leq 3kq2^{1/2}R_k(q, 0). \]

Defining \( T \) as in Lemmas 6.10, we obtain from (6.12.4), (6.12.6), (6.12.7) and Lemma 10

\[ |S| \leq \epsilon \sum \frac{p r^{r-1}}{k^r} \sum_{s=1}^1 \frac{p s^{r-1}}{k^s} \sum_{t=1}^1 \frac{p t^{r-1}}{k^t} S_r(p) \delta(q) e^{2\pi inq}. \]

Now \( q < 2^{\epsilon \lambda(n') - 1} \leq 2k \epsilon \lambda(n') - 1 \). Hence

\[ |S| \leq A_1(2k^{1/2} + 1) \phi(k+1) \left| \int_{\Omega_k} (S(q) \delta(q))e^{2\pi inq}dq \right|^2. \]
and the result follows, since $\frac{1}{\log t} - 3/(2\log 11) > \frac{1}{2}$ and $t < 32k^2 \log k$.

6.13. Lemma. If $(c)$ satisfies the conditions of Lemma 6.12 in an interval $[P + 1, P + N]$, where $N \leq Q$, and $\lambda \leq \lambda_0 \leq \lambda_1$, then

$$\sum_{a \leq t \leq a + T} \sum_{\chi} \frac{1}{\log x} < A_2 \log \log Q \log \log \log Q.$$  

(6.13.1)

If $\lambda \leq \lambda_0 \leq \lambda_1$, the conditions of the previous theorem are satisfied when $Q$ is replaced by $N$, and (6.13.2) follows at once from (6.12.2). On the other hand, if $\lambda_0 > \lambda_1$, then

$$\sum_{a \leq t \leq a + T} \sum_{\chi} \frac{1}{\log x} < N < \lambda \leq \lambda_1 \leq Q,$$

and (6.13.1) again follows.


$$\zeta(1 + it) = O((\log \log \log \log t)^{\frac{1}{2}}).$$

Let $$(1 + it)^{-1} \log x = \frac{1}{2\pi} \log \frac{1}{1 - 2\pi}.$$ Let $\alpha < x < \beta$. Since $(-1)^{x+y}x^2$ is steadily decreasing, we can divide the interval $(\alpha, \beta)$ into not more than $\beta + 1$ intervals, in each of which the inequality of the form (6.12.1) holds, where $\lambda$ depends on the particular interval, and satisfies

$$\frac{1}{2\pi} \log x^{\frac{1}{2}} a^{\frac{1}{2}} < \lambda \leq \frac{t}{2\pi} a^{\frac{1}{2}}.$$  

(6.14.1)

Let $w = \alpha, \beta, \log \alpha > \log \beta$, and

$$b = \left[ \frac{\log t}{\log \alpha} \right].$$

Then

$$\zeta(1 + it) = O((\log \log \log \log t)^{\frac{1}{2}}).$$

Clearly $\lambda \leq Q^{-d}$, while $\lambda > Q^{-d}$ if $Q > 2^{\pi^2} x^{\frac{1}{2}} (\log x)$, or if

$$\log x > \left( \log \frac{\log x}{\log \alpha} \right) \log 2 + \log \log \log \log t + \log \log x,$$

and this is true if $x$ is large enough. It follows from Lemma 6.13 that

$$\sum_{\alpha < \chi < \beta} \frac{1}{\log \chi} = O((\log \log \log \log t)^{\frac{1}{2}}),$$

where $\rho$ is defined as in § 6.12. Hence

$$\sum_{\alpha < \chi < \beta} \frac{1}{\log \chi} = O((\log \log \log \log t)^{\frac{1}{2}}).$$

(6.15.1)

Suppose that $k \log k < A \log k$, with a sufficiently small $A$, or

$$\log a > A(\log \log \log \log t)^{\frac{1}{2}}$$

with a sufficiently large $A$. Then

$$\sum_{\alpha < \chi < \beta} \frac{1}{\log \chi} = O((\log \log \log \log t)^{\frac{1}{2}}),$$

and the sum of $O(\log \log \log \log t)$ such terms is bounded.

Since $k \leq 7$, we also require that $k \leq \eta$. Using (6.10.1) with $r = 8$, and writing $\beta = \frac{1}{\theta} (\log \log \log \log t)^{\frac{1}{2}}$, we obtain

$$\zeta(1 + it) = \sum_{\chi} \frac{1}{\log \chi} + O(\log \log \log \log t)^{\frac{1}{2}}.$$

(6.15.1)

The last sum is bounded if

$$\log a = A(\log \log \log \log t)^{\frac{1}{2}}$$

with a suitable $A$, and the theorem follows.

6.15. If $0 < \alpha < 1$, we obtain similarly

$$\sum_{\alpha < \chi < \beta} \frac{1}{\log \chi} = O((\log \log \log \log t)^{\frac{1}{2}}),$$

and this is bounded if

$$\frac{1}{\log a} < A \log \log \log \log t$$

with a sufficiently small $A$. Hence in this region

$$\zeta(s) = O(\sum_{\alpha < \chi < \beta} \frac{1}{\log \chi} + O(1))$$

$$= \frac{1}{\log \alpha} + O(1),$$

and

$$\exp(A \log \log \log \log t), \frac{\log \beta}{(\log \log \log \log t)^{\frac{1}{2}}}.$$  

(6.15.1)

We can now apply Theorem 2.10, with

$$\rho(t) = A(\log \log \log \log t)^{\frac{1}{2}}, \quad \rho(t) = A \log \log \log \log t.$$

Hence there is a region

$$s \gg 1 - \frac{A}{\log \log \log \log t}$$

(6.15.1)
which is free from zeros of \( f(x); \) and by Theorem 3.11 we have also
\[
\frac{1}{\lambda(1 + i)} = O(\log \log \log \theta), \quad \frac{1}{\lambda(1 + i)} = O(\log \log \log \log \theta),
\]
(6.15.2), (6.15.3)

**NOTES FOR CHAPTER 6**

6.16. Further improvements have been made in the estimation of \( J(q, l). \) The most important of these is due to Karamata [2] who used a \( p \)-adic analogue of the argument given here, thereby producing a considerably simplification of the proof. Moreover, as was shown by Steckin [1], one is then able to sharpen Lemma 6.9 to yield the bound
\[
J(q, l) < C^{\delta^k(1-\delta)(1+\delta)(1+\delta)(1+\delta)},
\]
for \( l > kr, \) where \( k > 2, r > 1 \) is a positive integer, \( C \) is an absolute constant, and \( \delta = \frac{1}{2k}(1-1/k). \) Here one has a smaller value for \( \delta \) than formerly, but more significantly, the condition \( l > k^2 + \frac{1}{2}k + kr \) has been relaxed.

6.17. One can use Lemma 6.13 to obtain exponent pairs. To avoid confusion of notation, we take \( f(x) \) to be defined on \((a, h), \) with \( a < b < 2a \) and \( \lambda^{-1} < a < \lambda^{-1}. \) Then
\[
\hat{f}(x) \leq \lambda^{\hat{f}(x)} \leq A_k^{\hat{f}(x)} \leq A_k^{\hat{f}(x)} \log \theta.
\]

Now suppose that \((N, l, f, \lambda) \in \mathcal{F}(a, b) \) of (5.20), whence
\[
2N \lambda^{-x-a} \leq \lambda^{\gamma(N^{-1})} \leq 2N \lambda^{-x-a}
\]
with
\[
\lambda^{-x-a} = \gamma(N^{-1}) + \frac{1}{(k+1)!}
\]
We may therefore break up \( f(x) \) into \( O(s + a) \) subintervals \((a, b) \) with \( b \in \{1\} \times \ast \), on each of which one has
\[
\lambda^{-x-a} \leq \lambda \left| f(x+a) \right| \leq 2N,
\]
with \( \lambda = \frac{1}{2}\lambda^{-x-a}. \) We now choose \( k \) so that \( \lambda^{-1} < N < \lambda^{-1} \) for all \( a \) in the range \( N < a < 2N. \) To do this we take \( k \geq 7 \) such that
\[
\frac{N^k}{a_k} < \frac{1}{2} \frac{N^{-x-a}}{a_k},
\]
(6.17.1)

6.17. VINOGRADOV'S METHOD

Note that \( N^k / a_k \) tends to infinity with \( k, \) if \( N \geq 2, \) so this is always possible, providing that
\[
\frac{N^k}{a_k} < \frac{1}{2} \frac{N^{-x-a}}{a_k},
\]
(6.17.2)

The estimate (6.17.1) ensures that \( 2N < \lambda^{-1}, \) and hence, incidentally, that \( \lambda < 1. \) Moreover we also have
\[
N^k < \frac{1}{2} a_k \lambda^{-x-a} = \frac{1}{2} a_k \lambda^{-x-a} \frac{N^{-x-a}}{a_k},
\]
if \( N \geq 2^{x-a}, \) and so \( \lambda^{-1} < N. \) It follows that
\[
\sum_{\gamma(N^{-1})} \leq a_k \lambda^{\hat{f}(x)} \lambda^{-x-a} \log N
\]
(6.17.3)
for \( N \geq 2^{x-a}, \) subject to (6.17.2).

We shall now show that
\[
(n, q) = \left( \frac{1}{25} (m - 2m^2 \log m, 1 - \frac{1}{25} m \log m) \right)
\]
(6.17.4)

is an exponent pair whenever \( m > 2. \) If \( \gamma(N^{-1}) \geq 1 \) then \( \gamma \gamma(N^{-1}) \geq \gamma \gamma(N^{-1}) \) and the required bound (6.20.2) is trivial. If (6.17.4) fails, then \( \gamma(N^{-1}) \leq N \gamma(N^{-1}) \) and, using the exponent pair \((x, \gamma) = A^{\theta}(x, 1) \) (in the notation of (6.20.2) we have
\[
\sum_{\gamma(N^{-1})} \leq a_k \gamma(N^{-1}) \lambda^{-x-a} \lambda^{-x-a} \log N
\]
as required. We may therefore assume that \( \gamma(N^{-1}) < 1, \) and that (6.17.4) holds. Let us suppose that \( N > \max(2^{x-a}, 2(n, x + 1)). \) Then (6.17.1) yields
\[
N_{x-1} \leq \frac{5}{2} \frac{x+1}{2} \frac{x+2}{2} \frac{x+k-2}{k} \frac{N_{x-1}}{x+1}
\]
\[
\leq \left( \max \left( \frac{1}{2} \right) \right)^{x-1} \frac{N_{x-1}}{x+1} < 2(n, x+1)^{x-1} \frac{N_{x-1}}{x+1},
\]
whence
\[
\left( \frac{N}{x+1} \right)^{x-1} < 2(n, x+1)^{x-1}.
\]
Since \( N > 2(n, x+1)^{x-1} \) we deduce that \( k \leq m. \) Moreover we then have \( N \geq 2^{x-a} \geq 2^{x-a} \), so that (6.17.3) applies. Since \( k \) is bounded in
terms of $p$, $q$ and $s$, it follows that
\[
\sum_{n=1}^{N} \sigma(n) \psi(p^s) = \frac{N^{1-s} \log N}{\psi(p^s)} + \mathcal{O}(N^{\varepsilon})
\]
if $N \gg p^s$, and the required estimate (5.20.2) follows.

6.18. We now show that the exponent pair (0.17.4) is better than any pair $(a, b)$ obtainable by the $A$ and $B$ processes from (0.1), if $m \gg 10^a$. By this we mean that there is no pair $(a, b)$ with both $p \gg a$ and $q \gg b$. To do this we shall show that
\[
b + 5a^2 \geq 1.
\]
(6.18.1)
Then, since $3.525m^2 \log m < (m - 2)^2$ for $m \gg 10^a$, we have $q + 5a^2 < 1$, and the result will follow. Certainly (6.18.1) holds for $0, 1$. Thus it suffices to prove (6.18.1) by induction on the number of $A$ and $B$ processes needed to obtain $(a, b)$. Since $B(x, y) = (x, y)$ and $A(0, 1) = (0, 1)$, we may suppose that either $(a, b) = A_i(\delta, \beta)$ or $(a, b) = B_i(\delta, \beta)$, where $(\delta, \beta)$ satisfies (6.18.1). In the former case we have
\[
b + 5a^2 = \frac{7 + \delta}{2} + 5 \left( \frac{\delta}{2} + \frac{1}{2} \right) \geq \frac{7 + 2 - \delta}{2} + 5 \left( \frac{\delta}{2} + \frac{1}{2} \right) = 1
\]
for $0 < \gamma \leq 1$, and in the latter case
\[
b + 5a^2 = \frac{7 + \delta}{2} + 5 \left( \frac{\delta}{2} + \frac{1}{2} \right) \geq \frac{7 + 2 - \delta}{2} + 5 \left( \frac{\delta}{2} + \frac{1}{2} \right) = 1
\]
for $0 < \gamma < 1$. This completes the proof of our assertion.

The exponent pairs (0.17.4) are not likely to be useful in practice. The purpose of the above analysis is to show that Lemma 6.13 is sufficient for $f(x)$ to be approximated to any function for which the exponent pairs method can be used, and that there do exist exponent pairs not obtainable by the $A$ and $B$ processes.

6.19. Different ways of using $\psi(x)$ to estimate exponential sums have been given by Kurodov [1] and Vinogradov [1] (see Wallis [1; Chapter 2]) for an alternative expansion. These methods require more information about $f$ than a bound (6.18.1) for a single derivative, and so we shall give the results for partial sums of the $s$-function only. The two methods give qualitatively similar estimates, but Vinogradov's is slightly simpler, and is quantitatively better. Vinogradov's result, as given by Wallis [1], is
\[
\sum_{a < x \leq b} \sigma(x) = a^{1-s} + \mathcal{O}(a^{1-s})
\]
(6.19.1)
for $a < b \leq 2a$, $t \geq 1$, where
\[
t^{1-s} \leq a < t^{(s+1)/2},
\]
k > 1, and
\[
p = \frac{1}{600000}.
\]
The implied constant is absolute. Richert [3] has used this to show that
\[
\zeta(s) = \left( \log \log t \right)^{\frac{1}{2}},
\]
uniformly for $0 < \sigma < 2$, $t > 2$. The choices
\[
\theta(t) = \left( \log \log t \right)^{\frac{1}{2}}, \quad \varphi(t) = \log \log t
\]
in Theorems 6.10 and 6.11 therefore give a region
\[
\sigma \geq 1 - A \left( \log \log t \right)^{-\frac{1}{2}} - 1
\]
free of zeros, and in which
\[
\left( \frac{\psi(x)}{x} \right) \leq (\log \log t)^{\frac{1}{2}}, \quad \left( \frac{\varphi(x)}{x} \right) \leq \left( \log \log t \right)^{\frac{1}{2}}.
\]
The superiority of (6.19.1) over Lemma 6.13 lies mainly in the elimination of the term $\exp(300000 \log t)$, rather than in the improvement in the exponent $p$.

Various authors have reduced the constant 100 in (6.19.2), and the best result to date appears to be one in which 100 is replaced by 18.8 (Heath-Brown, unpublished).

6.19. We shall sketch the proof of Vinogradov's bound. The starting point is an estimate of the form (6.12.4), but with
\[
\frac{1}{x} \leq \frac{1}{x} \leq 2^{|x/a + \pi(x) - \pi(\alpha)|}
\]
(6.20.1)
in place of $T(x)$. One replaces $f(\pi - a) - f(x)$ by a polynomial
\[
T(x) = A_1 x + \ldots + A_k x^k
\]
as in §6.12, and then uses Hölder's inequality to obtain
\[
\left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right| \leq q^{t - 1} \sum_{\sigma} \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{t - 1} \sum_{\sigma} \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{t - 1}.
\]
\[
= q^{t - 1} \sum_{\sigma} n(\sigma_1, \ldots, \sigma_k) \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{t - 1}.
\]
where \(|\sigma| = 1, n(\sigma_1, \ldots, \sigma_k)\) denotes the number of solutions of
\[
u_1 + \ldots + \nu_k = \sigma_k \quad (1 \leq k \leq k),
\]
and
\[
G(\nu_1, \ldots, \nu_k) = A_1 \nu_1 e^{\mathbf{v}} + \ldots + A_k \nu_k e^{\mathbf{v}}.
\]
Now, by Hölder's inequality again, one has
\[
\left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{2t - 1} \leq q^{(t - 1)(2t - 1)} \left( \sum_{\sigma} n(\sigma_1, \ldots, \sigma_k) \right)^{2t - 2} \sum_{\mathbf{y}} \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{2t - 2}.
\]
Here
\[
\sum_{\sigma_1, \ldots, \sigma_k} n(\sigma_1, \ldots, \sigma_k) = q^t,
\]
and
\[
\sum_{\sigma_1, \ldots, \sigma_k} n(\sigma_1, \ldots, \sigma_k) = J(q, l).
\]
Moreover
\[
\sum_{\sigma_1, \ldots, \sigma_k} \sum_{\mathbf{y}} e^{2\pi i f(y)} = \sum_{\sigma_1, \ldots, \sigma_k} n^*(\tau_1, \ldots, \tau_k) \sum_{\sigma_1, \ldots, \sigma_k} e^{2\pi i f(y)},
\]
where
\[
H(\nu_1, \ldots, \nu_k; \tau_1, \ldots, \tau_k) = A_1 \nu_1 \tau_1 + \ldots + A_k \nu_k \tau_k,
\]
and \(n^*(\tau_1, \ldots, \tau_k)\) is the sum of \(\sigma(\nu) \ldots \sigma(\nu_k)\) subject to
\[
u_1 + \ldots + \nu_k = \nu_1 + \ldots + \nu_k = \tau_k \quad (1 \leq k \leq k).
\]

Since \(|n^*(\tau_1, \ldots, \tau_k)| \leq J(q, l)|\), it follows that
\[
\left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{2t - 1} \leq q^{(t - 1)(2t - 1)} \sum_{\sigma} \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{2t - 1} \sum_{\mathbf{y}} \min \left( \left| \sum_{\sigma} e^{2\pi i f(y)} \sigma_1 \tau_1 \right| \right)
\]
\[
\leq q^{(t - 1)(2t - 1)} \sum_{\sigma} \left| \sum_{\mathbf{y}} e^{2\pi i f(y)} \right|^{2t - 1} \sum_{\mathbf{y}} \min \left( \left| \sum_{\sigma} e^{2\pi i f(y)} \sigma \right| \right).
\]
At this point one estimates the sum over \(\tau_k\), getting a non-trivial bound whenever \(q^{1 - \epsilon} |A_<| \leq 1\). This leads to an appropriate result for the original sum (6.39.1), on taking \(i = \left| c \theta \right|^{t - 1}\) with a suitable constant \(c\). If we use Lemma 6.9, for example, to estimate \(J(q, l)\), then
\[\left(\frac{\theta}{\pi}\right)^{2t - 1} \leq 1.\]
One therefore sees that the implied constant in (6.39.1) is indeed independent of \(k\).
VII

MEAN-VALUE THEOREMS

7.1. The problem of the order of $\zeta(s)$ in the critical strip is, as we have seen, unsolved. The problem of the average order, or mean-value, is much easier, and, in its simplest form, has been solved completely. The form which it takes is that of determining the behaviour of

$$\frac{1}{T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt$$

as $T \to \infty$, for any given value of $s$. We also consider mean values of other powers of $\zeta(s)$.

Results of this kind have applications in the problem of the zeros, and also in problems in the theory of numbers. They could also be used to prove $O$-results if we could push them far enough; and they are closely connected with the $\Omega$-results which are the subject of the next chapter.

We begin by recalling a general mean-value theorem for Dirichlet series.

**Theorem 7.1.** Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

be absolutely convergent for $a > \sigma$, $b > \sigma$, respectively. Then for $a > \sigma$, $b > \sigma$,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(s+it)g(s-it) \, dt = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^\sigma}.$$  \hspace{1cm} (7.1.1)

For

$$f(s+it)g(s-it) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{b_n}{n^s} = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{2\sigma}} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\sigma} m^{\sigma}} e^{2\pi imt},$$

the series being absolutely convergent, and uniformly convergent in any finite $t$-range. Hence we may integrate term-by-term, and obtain

$$\frac{1}{2T} \int_{-T}^{T} f(s+it)g(s-it) \, dt = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{2\sigma}} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\sigma} m^{\sigma}} \frac{2\sin(T \log n/m)}{2T \log n/m}.$$  \hspace{1cm} (7.1.2)

The factor involving $T$ is bounded for all $T$, $m$, and $n$, so that the double series converges uniformly with respect to $T$, and each term tends to zero as $T \to \infty$. Hence the sum also tends to zero, and the result follows.

**7.1 MEAN-VALUE THEOREMS**

In particular, taking $b_n = a_n$ and $a = b = s$, we obtain

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \quad (a > \sigma).$$  \hspace{1cm} (7.1.3)

These theorems have immediate applications to $\zeta(s)$ in the half plane $a > 1$. We deduce at once

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt = \zeta(2a) \quad (a > 1),$$

and generally

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt = \frac{\zeta'(2a)}{2a} \quad (a > 1, \beta > 1).$$

Taking $a_n = d_n(s)$, we obtain

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt = \frac{\zeta'(2\sigma)}{2\sigma} \quad (a > 1).$$

By (1.2.10), the case $\beta = 2$ is

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(s+it)|^2 \, dt = \frac{\zeta'(2\sigma)}{2\sigma} \quad (a > 1).$$

The following sections are mainly concerned with the attempt to extend these formulæ to values of $a$ less than or equal to 1. The attempt is successful for $\sigma < 2$, only partially successful for $\sigma = 1$.

7.2. We require the following lemmas.

**Lemma.** We have

$$\sum_{n=1}^{\infty} \frac{1}{m^{\sigma} \log n/m} = O(T^{2\sigma-\sigma} \log T) \quad (7.2.1)$$

for $\frac{1}{2} < \sigma < 1$, and uniformly for $\frac{1}{2} < \sigma < \sigma_0 < 1$.

Let $\xi_n$ denote the sum of the terms for which $m < n$, $\Sigma_2$ the remainder. In $\Sigma_2$, $\log n/m > A$, so that

$$\xi_2 < A \sum_{n=1}^{\infty} m^{\sigma-\sigma_0} < A \left( \sum_{n=1}^{\infty} n^{\sigma-\sigma_0} \right)^{1/2} < A T^{1-\sigma_0}.$$  \hspace{1cm} (7.2.2)

In $\Sigma_2$ we write $m = n - r$, where $1 < r < n$, and then

$$\log n/m = -\log(1-r/n) > r/n.$$  \hspace{1cm} (7.2.3)

Hence

$$\xi_2 < A \sum_{n=1}^{\infty} \frac{(n-r)^\sigma r^{-\sigma}}{r/n} < A \sum_{n=1}^{\infty} \frac{n^{\sigma}}{r/n} \sum_{r=1}^{\infty} \frac{1}{r/n} < A T^{\sigma_0} \log T.$$  \hspace{1cm} (7.2.4)
Lemma. \[ \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} = O \left( \frac{\log^{1/2} T}{T} \right) \] (7.2.2)

Dividing up as before, we obtain
\[ \sum_{a=1} = O \left( \sum_{(a,n)=1} \frac{1}{n^{(n-1)/2} \log n/m} \right) = O(\log^{-1} T), \]
and
\[ \sum_{a} = O \left( \sum_{a=1} \frac{1}{n^{(n-1)/2} \log n/m} \right) = O(\log^{-1} T). \]

Theorem 7.2.
\[ \lim_{T \to \infty} \frac{1}{T} \sum_{\sigma > \frac{1}{2}} \sum_{a \in \mathbb{C}} \frac{1}{(s+i\sigma)^2} \frac{1}{t} \int_{-T}^{T} |Z(t)|^2 \, dt = \frac{\zeta(2\sigma)}{\sigma}. \]

We have already accounted for the case \( \frac{1}{2} < \sigma < 1 \), so that we now suppose that \( 1 < \sigma < \frac{1}{4} \). Since \( T \to \infty \), Theorem 4.11, with \( x = t \), gives
\[ \sum_{a=1} \frac{1}{n^{(n-1)/2} \log n/m} = O(t^{-1}). \]

say. Now
\[ \int_{-T}^{T} |Z(t)|^2 \, dt = \int_{-T}^{T} \left( \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \right) \, dt \]
\[ = \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \int_{-T}^{T} \left( \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \right) \, dt \]
\[ = \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \int_{-T}^{T} \left( \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \right) \, dt \]
\[ = T \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \int_{-T}^{T} \left( \sum_{a=1\atop \gcd(a,n)=1}^{n} \frac{e^{2\pi i ax/n}}{n^{(n-1)/2} \log n/m} \right) \, dt \]

provided that \( \sigma < \frac{1}{4} \). If \( T = 1 \), we can replace the \( \sigma \) of the last two terms by \( 1 \) say. In either case
\[ \int_{-T}^{T} |Z(t)|^2 \, dt \sim T(\zeta(2\sigma)). \]

Hence
\[ \int_{-T}^{T} |Z(t)|^2 \, dt = \int_{-T}^{T} |Z(t)|^2 \, dt + O \left( \int_{-T}^{T} |Z(t)|^2 \, dt \right) + O \left( \int_{-T}^{T} |Z(t)|^2 \, dt \right) \]
\[ = \int_{-T}^{T} |Z(t)|^2 \, dt + O \left( \int_{-T}^{T} |Z(t)|^2 \, dt \right) + O(\log T) \]
\[ = \int_{-T}^{T} |Z(t)|^2 \, dt + O(\log T) T \]
and the result follows.

7.2 Mean-Value Theorems

It will be useful later to have a result of this type which holds uniformly in the strip. It is

Theorem 7.2 (a).
\[ \int_{-T}^{T} |Z(t)|^2 \, dt \leq A T \min \left\{ \log T, \frac{1}{\sigma - 1} \right\} \]
uniformly for \( \frac{1}{4} < \sigma < 2 \).

Suppose first that \( \frac{1}{4} < \sigma < 1 \). Then we have, as before,
\[ \int_{-T}^{T} |Z(t)|^2 \, dt < T \sum_{a \in \mathbb{C}} n^{-1} \frac{1}{\zeta(2\sigma)} + O(T^{-1}(\log T)^2) \]
uniformly in \( \sigma \). Now
\[ \sum_{a \in \mathbb{Z}} n^{-3\sigma} < \sum_{n \in \mathbb{Z}} n^{-1} \leq A \log T \]
and also
\[ \sum_{n \in \mathbb{Z}} n^{-3\sigma} < \sum_{n \in \mathbb{Z}} n^{-1} \leq A \log T \]

Similarly
\[ T^{-1}(\log T)^2 \leq \frac{1}{T \sigma} \cdot \frac{1}{\zeta(2\sigma)}. \]

This gives the result for \( \sigma < \frac{1}{4} \), the term \( O(\log T) \) being dealt with as before.

If \( \frac{1}{4} < \sigma < 2 \), we obtain
\[ \int_{-T}^{T} |Z(t)|^2 \, dt < T \sum_{a \in \mathbb{C}} n^{-1} \frac{1}{\zeta(2\sigma)} + O(T^{-1}(\log T)^2), \]
and the result follows at once.

7.3. The particular case \( \sigma = \frac{1}{4} \) of the above theorem is
\[ \int_{-T}^{T} |Z(t)|^2 \, dt \sim T \log T. \]
We can improve this \( O \)-result to an asymptotic equality. But Theorem 4.11 is not sufficient for this purpose, and we have to use the approximate functional equation.

Theorem 7.3. As \( T \to \infty \)
\[ \int_{-T}^{T} |Z(t)|^2 \, dt \sim T \log T. \]

\[ \dagger \text{Littlewood (4).} \]
\[ \dagger \text{Hardy and Littlewood (2), (4).} \]
In the approximate functional equation (4.12.4), take $s = \frac{1}{2}, \gamma > 2$, and $x = t(2\pi n \log t), y = v \log t$. Then, since $x(1+\delta) = O(1)$,

$$\sum_{n \leq x} n^{-s} = \sum_{n \leq x} n^{-1+\delta} + O\left(\sum_{n \leq x} n^{-1}\right) + O(v \log x) + O(v^{-\delta})$$

say. Since

$$\sum_{n \leq x} n^{-1} \log n = O(T \log T) = o(T \log T),$$

it is, as in the proof of Theorem 7.2, sufficient to prove that

$$\int_0^T \frac{1}{n} \sum_{\sigma \leq n} n^{-1} \frac{1}{n^s} dt \sim T \log T.$$

Now

$$\int_0^T \frac{1}{n} \sum_{\sigma \leq n} n^{-1} \frac{1}{n^s} dt = \int_0^T \sum_{\sigma = 1}^T \sum_{\sigma \leq n} \frac{1}{\sigma^s} dt,$$

In inverting the order of integration and summation, it must be remembered that $\sigma$ is a function of $t$. The term in $(m, n)$ occurs if $x > \max(m, n) = T/(2\pi n \log T)$ say, where $T = T(m, n)$. Hence, writing $X = T/(2\pi n \log T)$,

$$\int_0^T \frac{1}{n} \sum_{\sigma \leq n} \frac{1}{\sigma^s} dt = \sum_{r > 0} \frac{1}{r^s} \int_0^X \frac{1}{n^s} dt$$

The first term is

$$T \log T + O(T \log T).$$

The second term is

$$O\left(\sum_{n \leq x} \log n\right) = O(X \log X) = O(T),$$

and, by the first lemma of § 7.2, the last term is

$$O(X \log X) = O(T \log T).$$

This proves the theorem.

7.4. We shall next obtain a more precise form of the above mean-value formula.†

**Theorem 7.4.**

$$\int_0^T \left[\left(1+\frac{t}{\log x}\right) \frac{1}{n} \sum_{\sigma \leq n} \frac{1}{\sigma^s} \right] dt = \frac{T}{2\pi} \log T + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} ds + O(T \log T).$$

We first prove the following lemma.

**Lemma.** If $n < T/2\pi$,

$$\int_{-\infty}^{\infty} \frac{T}{2\pi} \log T + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} ds$$

If $n > T/2\pi, c > \frac{1}{2}$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{T}{2\pi} \log T + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} ds$$

We have

$$\frac{1}{\Gamma(1-s)} = 2^{1-s} \cos \frac{\pi s}{2} \sum_{n=0}^{\infty} \frac{1}{n!(n+1-s)}$$

This has poles at $s = -2v (v = 0, 1, ...)$ with residues

$$\frac{(-1)^v 2^{1+2v} \pi}{(2v)!}.$$

Also, by Stirling's formula, for $-\pi + \delta < \arg(-s) < \pi - \delta$

$$\Gamma(1-s) = \frac{2\pi}{\sin \pi s} \left(1+O\left(\frac{1}{|s|}\right)\right).$$

The calculus of residues therefore gives

$$\frac{1}{2\pi} \int_{1-2s}^{1-2s} \int_{-\infty}^{\infty} \frac{T}{2\pi} \log T + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} ds$$

Also, since

$$\frac{1}{\Gamma(1-s)} = O(e^{\pi^2}),$$

† Ingham (1) obtained the error term $O(T \log T)$; the method given here is due to Atkinson (1).
\[ \int_{-\infty}^{T} x(1-s)x^{-1}\, ds = o \left( \frac{1}{\sqrt{T}} \right) \quad \text{as} \quad T \to \infty. \]

Similarly for the integral over \((T, \infty)\) and \((0, T/2c)\).

Again, for a fixed \(a\),
\[ x(1-s) = \left( \frac{\pi}{2} \right) e^{-a} - o(1) \quad (T \gg 1). \]

Hence
\[ \int_{-\infty}^{T} x(1-s)x^{-1}\, ds = \int_{-\infty}^{T} e^{-a} - o(1) \, ds = O(1). \]

where
\[ F(s) = \int_{-\infty}^{s} x(1-t)\, dt = \log T - T/2c. \]

Hence by Lemma 4.3, the last integral is of the form
\[ O \left( \frac{1}{\log T} \right) \]

uniformly with respect to \(T\). Taking, for example, \(T = 2T > 4\pi x\), we obtain (4.4.2). Again
\[ \int_{-\infty}^{T} x(1-s)x^{-1}\, ds = \int_{-\infty}^{T} e^{-a} - o(1) \, ds + O(1). \]

and (4.3.4) follows from Lemma 4.3.

In proving (4.3.1) we may suppose that \(T/2\pi\) is half an odd integer; for a change of \(O(1)\) in \(T\) alters the left-hand side by \(O(1/T)\), since \(2(1+i) = O(1/T)\), and the leading terms on the right-hand side by \(O(1/T)\). Now the left-hand side is
\[ \frac{1}{2\pi} \left( \frac{T}{2\pi} \right)^{1/2} dt = \frac{1}{2\pi} \left( \frac{T}{2\pi} \right)^{1/2} dt. \]

and (4.3.4) follows from Lemma 4.3.

By (7.4.3),
\[ L_1 = 2\pi \sum_{n \in T/\pi} \frac{d(n)}{n} + O \left( \sum_{n \in T/\pi} \frac{d(n)}{n} \right) = \log T \sum_{n \in T/\pi} \frac{d(n)}{n}. \]

The first term is
\[ 2\pi \left( \frac{T}{2\pi} \log \frac{T}{2\pi} + (2\pi - 1) \frac{T}{2\pi} - O(T) \right) \]

and (7.4.4) follows from this.

The last term is also closely of this form. Hence
\[ L_1 = T \log T - 2\pi - 1 - \log 2\pi + O(T^{-1}). \]

Next, if \(\varepsilon > 0\),
\[ L_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \right)^{1/2} \frac{d(n)}{n} \, dn + \int_{-\infty}^{\infty} \frac{d(n)}{n} \left( x(1-x) \frac{d(n)}{n} - A \right) \]

A being the residue of \(\Psi(1-x)\) at \(s = 1\).

Since \(\Psi(x) = O(x^{-1})\), and \(\Psi(x) + \Psi(x)\) and \(\sum d(n)x^{-1}\) are both of the form \(O(x^{-\varepsilon})\) \((\varepsilon > 1)\), the first term is
\[ \sum_{n \in T/\pi} \frac{d(n)}{n^2} \left( \log(2\pi/T) + 1 \right) \]

the second term is
\[ O \left( \sum_{n \in T/\pi} \frac{d(n)}{n^2} \right) = \log T \sum_{n \in T/\pi} \frac{d(n)}{n^2}. \]

Since \(c\) may be as near to 1 as we please, this proves the theorem.

A more precise form of the above argument shows that the error-term in (7.4.1) is \(O(T\log T)\). But a more complicated argument.

\textit{Footnote:}

\textit{1} See [12.1], or Hardy and Wright, "An Introduction to the Theory of Numbers, Theorem 32.

\textit{2} Ibid. Theorem 35.

\textit{3} Ibid. Theorem 37.
depending on van der Corput’s method, shows that it is $O(T^{2\log^2 T})$, and presumably further slight improvements could be made by the methods of the later sections of Chapter V.

7.5. We now pass to the more difficult, but still manageable, case of $|Z(t)|^4$. We first prove

**Theorem 7.5.**

$$
\lim_{T \to \infty} \frac{1}{T} \int_{(a-1)}^{(a+1)} |Z(t+it)|^4 \, dt = \frac{C_2}{C_4} \quad (\varepsilon > 0).
$$

Take $x = y = \sqrt{2}T$ and $\varepsilon > \frac{1}{4}$ in the approximate functional. We obtain

$$
Z(t) = \sum_{n < \sqrt{2}T} \frac{1}{n^{rac{1}{2}}} + O\left(\frac{1}{T}\right) = \mathcal{L}_2 + O\left(\frac{1}{T}\right),
$$

say. Now

$$
|Z(t)|^4 = \sum_{n < \sqrt{2}T} \frac{1}{n^2} + \sum_{n < \sqrt{2}T} \frac{1}{n^2} \frac{1}{T^2} + \sum_{n < \sqrt{2}T} \frac{1}{n^2} \frac{1}{T^2} + \cdots
$$

(7.5.1)

where each variable runs over $(1, \sqrt{2}T)$. Hence

$$
\int |Z(t)|^4 \, dt = \int \frac{1}{n^2} \mathcal{K}_2 \, \frac{1}{n^2} \, dt
$$

$$
= \sum_{n \mathcal{K}_2} \frac{1}{(n \mathcal{K}_2)^2} \int \left(\frac{1}{n \mathcal{K}_2}\right)^2 \, dt,
$$

where $\mathcal{K}_2 = 2\varepsilon \max(n^2, a^2, b^2, c^2) - \cdots$.

The number of solutions of the equations $n = \frac{1}{2} r^2 + \cdots$ if $r < \sqrt{2}T(2\varepsilon)$, and in any case does not exceed $d(r) = \varepsilon$. Hence

$$
\sum_{n < \sqrt{2}T} \frac{1}{(n \mathcal{K}_2)^2} = T \sum_{r < \sqrt{2}T(2\varepsilon)} \frac{d(r)^2}{r^2} + O\left(\frac{1}{T}\right) \sum_{r < \sqrt{2}T(2\varepsilon)} \frac{d(r)^2}{r^2}
$$

$$
= T \sum_{r < \sqrt{2}T(2\varepsilon)} \frac{d(r)^2}{r^2} - T C_2 \frac{C_2}{C_4},
$$

(7.5.2)

† Hardy and Littlewood (4).

7.6. The problem of the mean value of $|\xi(t+it)|^4$ is a little more difficult. If we follow out the above argument, with $\varepsilon = \frac{1}{4}$ as accurately as possible, we obtain

$$
\int |\xi(t+it)|^4 \, dt = O(T \log^2 T),
$$

(7.6.1)
but fail to obtain an asymptotic equality. It was proved by Ingham\(^\dagger\) by means of the functional equation for \(\zeta(s)\) that

\[
\int_1^T \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{T \log^2 T}{2\pi},
\]

(7.6.2)

The relation

\[
\int_1^T \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{T \log^2 T}{2\pi}
\]

(7.6.3)

is a consequence of a result obtained later in this chapter (Theorem 7.16).

7.7. We now pass to still higher powers of \((s)\). In the general case our knowledge is very incomplete, and we can state a mean-value formula in a certain restricted range of values of \(\sigma\) only.

**Theorem 7.7.** For every positive integer \(k \geq 2\)

\[
\lim_{T \to \infty} \frac{1}{T} \int_1^T \left| \Gamma \left( \frac{1}{2} + it \right) \right|^{2k} dt = \sum_{n \leq T^{1/2}} \frac{d(n)}{n^{\sigma}} \left( \sigma > 1 - \frac{1}{k} \right).
\]

(7.7.1)

This can be proved by a straightforward extension of the argument of \(\S\) 5.8. Starting again from (7.3.1), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_1^T \left| \Gamma \left( \frac{1}{2} + it \right) \right|^{2k} dt = \sum_{n \leq T^{1/2}} \frac{d(n)}{n^{\sigma}} \left( \sigma > 1 - \frac{1}{k} \right)
\]

(7.7.1)

where each variable runs over \(\{1, \ldots, T/2\}\). The leading term goes in the same way as before, \(d(n)\) being replaced by \(d(n)\). The main \(O\)-term is

\[
O \left( \sum_{n \leq T^{1/2}} \frac{1}{n^{\sigma}} \cdot \log n \right) = O \left( T^{1 - \sigma} \right).
\]

The corresponding term in

\[
\int \left| \Gamma \left( \frac{1}{2} + it \right) \right|^{2k} dt
\]

is

\[
O \left( T^{1 - \sigma} \right)
\]

and since \(\Gamma^{(n)} = O \left( n^{-\sigma + \epsilon} \right)\), we obtain \(O \left( T^{1 - \sigma + \epsilon} \right)\) again. These terms are \(O\) if \(\sigma > 1 - 1/k\), and the theorem follows as before.

7.8. It is convenient to introduce at this point the following notation. For each positive integer \(k\) and each \(\alpha\), let \(\mu_\alpha(\sigma)\) be the lower bound of positive numbers \(\xi\) such that

\[
\int_1^T \left| \Gamma \left( \frac{1}{2} + it \right) \right|^{2k} dt = O \left( T^\alpha \right).
\]

\(\dagger\) Ingham (1).
Writing

\[ F(T) = \int_0^T f(a + iT)^s \, dt \leq C(T^s + 1) \]

we have by Stirling's theorem (with various values of \( K \))

\[
I(a) < K \int_0^T f(0) t \vartheta(t^s - 1) t^{s - 1/2} dt
\]

\[
= K \int_0^T f(0) (t^{(s-1)} + 1)(t^{(s-1)} - 1) t^{s - 1/2} dt
\]

\[
< KC \int_0^T (t^{s-1} + 1) t^{s-1/2} dt
\]

\[
< KC \int_0^T \left[ \left( \frac{1}{n} \right)^{s-1} + 1 \right] e \vartheta \, dt
\]

\[
< C K (n^{s-1} + 1) < C K (n^{s-1} + 1)^{1/2}
\]

Similarly for \( I(b) \). Hence

\[ I(a) < K(3 + a - 1) e^{n \vartheta} \left( \frac{1}{n} \right)^{s-1} e^{n \vartheta} \left( \frac{1}{n} \right)^{s-1} + 1 \]

Also

\[ I(a) < K \int_0^T f(0 + iT)^s \, dt > K \int_0^T e^{n \vartheta(t^s - 1) t^{s - 1/2} dt} \]

Putting \( \delta = 1/T \), the result follows.

If \( f(t) \) has a pole of order \( k \) at \( s_k \), we argue similarly with \( (a - s_k)^{k/2} \); this merely introduces a factor \( T^k \) on each side of the result, so that (7.8.3) again follows.

Replacing \( T \) in (7.8.3) by \( T^{-1}, T^{-}, \ldots \), adding, we obtain the result:

\[ \int_0^T f(a + iT)^s \, dt = O(T^s), \quad \int_0^T f(b + iT)^s \, dt = O(T^s), \]

then

\[ \int_0^T [f(a + iT)]^s \, dt = O(T^{s+1}(a-b)\vartheta(a-b) + 1) \]

Taking \( f(s) \) = \( \vartheta(s) \), the convexity of \( \mu_2(s) \) follows.

7.9. An alternative method of dealing with these problems is due to Carlson.† His main result is

**Theorem 7.9.** Let \( \alpha \) be the lower bound of numbers \( a \) such that

\[
\int_0^T [f(a + iT)^s \, dt = O(1) \tag{7.9.1}
\]

Then

\[
\alpha = \max \left( 1 - \frac{1}{1 + \mu_2(a)}, 1 \right)
\]

for \( 0 < a < 1 \).

We first prove the following lemma.

**Lemma.** Let \( f(t) = \sum_{n=1}^\infty a_n \) be absolutely convergent for \( s > 1 \). Then

\[
\sum_{n=1}^\infty a_n \frac{t^s}{n^{s-1}} = \frac{1}{2\pi i} \int_{C(+i\infty)} \Gamma(s-z)f(w)w^{s-1} dw
\]

for \( \delta > 0 \), \( c > 1 \), \( \epsilon > c \).

For the right-hand side in

\[
\frac{1}{2\pi i} \int_{C(+i\infty)} \Gamma(s-z)w^{s-1} dw = \sum_{n=1}^\infty a_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s-z)/(n)^{s-1} dw
\]

\[
= \sum_{n=1}^\infty a_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w)/(n)^{s-1} dw
\]

The inversion is justified by the convergence of

\[
\int_0^\infty \Gamma(s-z)/(n)^{s-1} dw
\]

Taking \( a_n = d_n(s), f(s) = \vartheta(s), c = 2 \), we obtain

\[
\sum_{n=1}^\infty d_n(s) e^{n} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s-z)/(n)^{s-1} dw \quad (s < 2)
\]

Moving the contour to \( R(s) = \alpha \), where \( \alpha - 1 < a < \alpha \), we pass the pole of \( \Gamma(w-s) \) at \( w = s \), with residue \( \vartheta(s) \), and the pole of \( \Gamma(w) \) at \( w = 1 \), where the residue is a finite sum of terms of the form

\[
K_{a_n}(\Gamma(1-s)\vartheta(s-1))
\]

† Carlson (3), (3).
This residue is therefore of the form \( O(\delta^{n-c+1} e^{-\delta n}) \), and, if \( \delta > |\tau|^{-1} \), it is of the form \( O(e^{-\delta n}) \). Hence

\[
U(s) = \sum_{\nu=1}^{\infty} \frac{d_\nu(s)}{n^\nu} e^{-\nu s} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(u-i)Q(w)e^{-w} dw + O(e^{-\delta n}).
\]

Let us call the first two terms on the right \( Z_1 \) and \( Z_2 \). Then, as in previous proofs, if \( \sigma > \frac{1}{4} \),

\[
\begin{align*}
\int_{-\infty}^{\infty} |Z_1|^2 dt &= O\left(T \sum \frac{d_\nu(\nu)}{\nu^{n\sigma}}\right) + O\left(\sum \frac{d_\nu(\nu)}{\nu^{n\sigma}[\log \nu]^n}\right) \\
&= O(T) + O\left(\sum \frac{d_\nu(\nu)}{\nu^{n\sigma}[\log \nu]^n}\right) \\
&= O(T) + O(T^{1/2}).
\end{align*}
\]

by (7.2.2). Also, putting \( u = \sigma + iv \),

\[
|Z_2| \leq \frac{3^{n-1}}{2\pi} \int_{-\infty}^{\infty} \left| \Gamma(u+it) \right|^{1/2} dt \leq \frac{3^{n-1}}{2\pi} \left( \int_{-\infty}^{\infty} \left| \Gamma(u+it) \right|^2 dt \right)^{1/2}.
\]

The first integral is \( O(1) \), while for \( |t| \ll T \)

\[
\left( \int_{-T}^{T} + \int_{T}^{+T} \right) \left| \Gamma(u+it) \right|^{2} dt \leq \left( \int_{-T}^{T} + \int_{T}^{+T} e^{2\pi^2 t^2} |t|^2 dt \right) \leq O(e^{-\delta T}).
\]

Hence

\[
\begin{align*}
\int_{-\infty}^{\infty} |Z_2|^2 dt &= O\left(3^{n-1} \int_{-\infty}^{\infty} \left| \Gamma(u+it) \right|^2 dt \right) \leq O\left(3^{n-1} T^{1/2} \right) \\
&= O\left(3^{n-1} \int_{-\infty}^{\infty} \left| \Gamma(u+it) \right|^2 dt \right) \leq O\left(3^{n-1} T^{1/2} \right).
\end{align*}
\]

Hence

\[
\int_{-\infty}^{\infty} |Z_2|^2 dt = O(T) + O(T^{1/2}) + O(T^{1/2}).
\]

Let \( \delta = 2 + \sum_{\nu=1}^{\infty} \frac{d_\nu(s)}{n^\nu} e^{-\nu s} \), so that the last two terms are of the same order, apart from \( \epsilon \). These terms are then \( O(T) \) if

\[
\sigma > \frac{1}{4} = \frac{1-\epsilon}{1+\mu_3(s)}
\]

For such values of \( \sigma \), replacing \( T \) by \( \frac{1}{T} \), \( \frac{1}{T} \), and adding, it follows that (7.9.1) holds. Hence \( \mu_3 \) is less than any such \( \varphi \), and the theorem follows.

A similar argument shows that, if we define \( \varphi_2 \) to be the lower bound of numbers \( \sigma \) such that

\[
\frac{1}{2\pi} \int_{T}^{+T} |\zeta(u+it)|^{2} dt = O(T^2),
\]

then actually \( \varphi_2 = \varphi \). For clearly \( \varphi_2 \leq \varphi \), and the above argument shows that, if \( \sigma > \varphi_2 \), and \( \sigma < \varphi \), then

\[
\frac{1}{2\pi} \int_{T}^{+T} |\zeta(u+it)|^{2} dt = O(T^2) + O(T^{1/2} + \sigma + T^{1/2}).
\]

Taking \( \delta = T^{-3}, \) where \( 0 < \lambda < 1/(2-2\alpha) \), the right-hand side is \( O(T) \). Hence \( \xi_2 < \lambda \), and so \( \varphi_3 \leq \varphi \).

It is also easily seen that

\[
\frac{1}{2\pi} \int_{T}^{+T} |\zeta(u+it)|^{2} dt \sim \sum_{\nu=1}^{\infty} \frac{d_\nu(\nu)}{\nu^{10}} \left( \sigma > \varphi_2 \right).
\]

For the term \( O(T) \) of the above argument is actually

\[
\frac{1}{2\pi} \sum \frac{d_\nu(\nu)}{\nu^{10}} |\zeta(\nu)|^{2} dt = \frac{1}{2\pi} \sum \frac{d_\nu(\nu)}{\nu^{10}} + c(T),
\]

and the result follows by obvious modifications of the argument. This is a case of a general theorem on Dirichlet series.

Theorem 7.9 (A). If \( \mu(s) \) is the \( \mu \)-function defined in § 5.1,

\[
1 - \epsilon = \frac{1}{1+2\mu(\sigma)},
\]

for \( k = 1, 2, \ldots \).

Since \( \Gamma(u+it) = O(e^{\pi|t|}) \),

\[
\frac{1}{2\pi} \int_{T}^{+T} |\zeta(u+it)|^{2} dt = O(T^{1/2} \int_{T}^{+T} |\zeta(u+it)|^{2} dt),
\]

and hence

\[
\mu_3(\epsilon) = 2\mu_3(\sigma) + \mu_3(\epsilon).
\]

Since \( \mu_3(\sigma) = 0 \), this gives \( \mu_3(\sigma) = 0 \), and the result follows.

1 See E. C. Titchmarsh, Theory of Functions, § 9.21.
These formulae may be used to give alternative proofs of Theorems 7.2, 7.5, and 7.7. It follows from the functional equation that

$$\mu_2(1 - s) = \mu_2(s) + 2\xi(1 - s).$$

Since $\mu_2(s) = 0$, $\mu_2(1 - s) > 0$, it follows that $\alpha_2 > \frac{1}{2}$. Hence, putting $s = 1 - \alpha_2$ in Theorem 7.8, we obtain either $\alpha_2 = \frac{1}{2}$ or

$$\alpha_2 < \frac{1}{2} + \frac{\alpha_2}{\alpha_2 - 1},$$

i.e.,

$$2\alpha_2 - 1 < \frac{2\alpha_2 - 1}{\alpha_2 - 1}(1 - \alpha_2).$$

Hence $\alpha_2 = \frac{1}{2}$, or

$$0 < \alpha_2 < \frac{1}{2}, \quad 1 < k(1 - \alpha_2), \quad \alpha_2 < \frac{1}{k}.$$

(7.9.3)

For $k = 2$ we obtain $\alpha_2 = \frac{1}{2}$, but for $k > 2$ we must take the weaker alternative (7.9.3).

7.10. The following refinement† on the above results uses the theorems of Chapter V on $\mu(r)$.

THEOREM 7.10. Let $k$ be an integer greater than 1, and let $n$ be determined by

$$n = (v - 1)^{2k - 1} + 1 < v^{2k - 1} + 1.$$  \hspace{1cm} (7.10.1)

Then

$$\alpha_2 < \frac{1}{2} + \frac{1}{2k - 1}.$$  \hspace{1cm} (7.10.2)

The theorem is true for $k = 2$ ($v = 1$). We then suppose it true for all $l$ with $1 < l < k$, and deduce it for $k$.

Take $l = (v - 1)^{2k - 4} + 1$, where $v$ is determined by (7.10.1). Then $\mu_2(s) = 0$, provided that

$$s > 1 - \frac{1}{2k - 3} = \frac{1}{2k - 1}.$$  \hspace{1cm} (7.10.3)

Taking $s = 1 - 2^{k-1} + \epsilon$, we have, since

$$\int_1^\infty \left| \left( \zeta(s + it) \right)^{1/2} \right|^2 d\epsilon \leq \max_{1 < l < k} \left| \left( \zeta(s + it) \right)^{l/2} \right|^2 d\epsilon,$$

$$\mu_2(s) \leq 2(2k - 1)\alpha_2(1 + \epsilon) + \mu_2(s)$$

$$= 2(2k - 1)\alpha_2(1 + \epsilon)$$

$$\leq \frac{2(2k - 1)^{2k-1} - 1}{(v + 1)^{2k-4} - 1}.$$  \hspace{1cm} (7.10.4)

† Davenport (1), Hasse grove (1).

7.10.1. Mean-Value Theorems

by Theorem 5.8. Hence, by Theorem 7.9,

$$\alpha_2 < 1 - \frac{1}{2k - 1} \frac{1}{(v + 1)^{2k-4}} = 1 - \frac{1}{2k - 1}.$$  \hspace{1cm} (7.10.5)

The theorem therefore follows by induction.

For example, if $k = 3$, then $v = 2$, and we obtain

$$\alpha_2 < \frac{1}{3}$$

instead of the result $\alpha_2 < \frac{1}{2}$ given by Theorem 7.7.

7.11. For integral $k$, $d_k(n)$ denotes the number of decompositions of $n$ into $k$ factors. If $k$ is not an integer, we can define $d_k(n)$ as the coefficient of $n^{-s}$ in the Dirichlet series for $\zeta(s)$, which converges for $s > 1$.

We can now extend Theorem 7.7 to certain non-integer values of $k$.

THEOREM 7.11. For $0 < k < 2$

$$\lim_{s \rightarrow 2} \int \frac{1}{\zeta(s + it)^{k/2}} d\epsilon = \sum_{n=1}^\infty \frac{d_k(n)}{n^s}.$$  \hspace{1cm} (7.11.1)

This is the formula already proved for $k = 1$, $k = 2$; we now take $0 < k < 2$. Let

$$\zeta_k(s) = \prod_{p \neq 2} \frac{1}{1 - p^{-s}}.$$  \hspace{1cm} (7.11.2)

The proof depends on showing (i) that the formula corresponding to (7.11.1) with $\zeta_k$ instead of $\zeta$ is true; and (ii) that $\zeta_k(s)$, though it does not converge to $\zeta(s)$ for $s < 1$, still approximates to it in a certain average sense in this strip.

We have, if $A > 0$,

$$\left| n \right|^{1/2} - \sum_{n=1}^\infty \frac{d_k(n)}{n^s}$$

say, where the series on the right converges absolutely for $s > 0$, and $d_k(n) = d_k(n)$ if $n < N$, and $0$ otherwise. Hence

$$\lim_{s \rightarrow 2} \int \frac{1}{\zeta_k(s + it)^{k/2}} d\epsilon = \sum_{n=1}^\infty \frac{\left| d_k(n) \right|^k}{n^s}.$$  \hspace{1cm} (7.11.3)

and

$$\lim_{s \rightarrow 2} \int \frac{1}{\zeta_k(s + it)^{k/2}} d\epsilon = \sum_{n=1}^\infty \frac{\left| d_k(n) \right|^k}{n^s}.$$  \hspace{1cm} (7.11.4)

† Ingham (4); proof by Davenport (1).
We shall next prove that
\[
\lim_{N \to \infty} \int_{-T}^{T} \left| \mathcal{L}(\sigma + it) - \mathcal{L}(\sigma - it) \right|^2 dt = 0 \quad (\sigma > \frac{1}{2}).
\]  
(7.11.4)

By Hölder's inequality,
\[
\int_{-T}^{T} \left| \mathcal{L}(\sigma + it) - \mathcal{L}(\sigma - it) \right|^2 dt \leq \int_{-T}^{T} \left| \mathcal{L}(\sigma + it) \right|^2 dt \int_{-T}^{T} \left| \mathcal{L}(\sigma - it) \right|^2 dt^{\frac{1}{2}}.
\]  
(7.11.5)

Now \( \{n \xi(s) \} \) is regular everywhere except for a pole at \( \sigma = 1 \), and is of finite order in \( t \). Also, for \( \sigma > \frac{1}{2} \),
\[
\int_{-T}^{T} \left| \mathcal{L}(\sigma + it) \right|^2 dt \leq \int_{-T}^{T} (1 + 2T^2) d\xi(s)^2 dt = O(T).
\]

Hence, by a theorem of Caron,\(^\dagger\)
\[
\lim_{N \to \infty} \int_{-T}^{T} \left| n \xi(s + it) - n \xi(s - it) \right|^2 dt = \sum_{n \geq 1} \rho(s, n)^2
\]
for \( \sigma > \frac{1}{2} \), where \( \rho(s, n) \) is the coefficient of \( n^{-s} \) in the Dirichlet series of \( \{n \xi(s) \} \). Now \( \rho(s, n) = 0 \) for \( n \leq N \), and \( 0 \leq \rho(s, n) \ll \beta(n) \) for all \( n \). Since \( \sum n \beta(n) e^{\nu n} \) converges, it follows that
\[
\lim_{N \to \infty} \int_{-T}^{T} \left| n \xi(s + it) - n \xi(s - it) \right|^2 dt = 0
\]  
(7.11.6)

(7.11.4) now follows from (7.11.5), (7.11.6), and (7.11.3).

We can now deduce (7.11.1) from (7.11.3) and (7.11.4). We have
\[
\left\{ \int_{-T}^{T} |\mathcal{L}(\sigma + it)|^2 dt \right\}^2 \leq \left\{ \int_{-T}^{T} \left| \mathcal{L}(\sigma + it) \right|^2 dt \right\} \left\{ \int_{-T}^{T} \left| \mathcal{L}(\sigma - it) \right|^2 dt \right\}^{\frac{1}{2}},
\]
where \( R = 1 \) if \( 0 < 2k < 1 \), \( R = 1/2k \) if \( 2k > 1 \). Similarly
\[
\left\{ \int_{-T}^{T} |\mathcal{L}(\sigma + it)|^2 dt \right\} \leq \left\{ \int_{-T}^{T} |\mathcal{L}(\sigma + it) \right|^2 dt \right\} + \left\{ \int_{-T}^{T} |\mathcal{L}(\sigma - it) \right|^2 dt \right\},
\]
and (7.11.1) clearly follows.

\(\dagger\) See Titchmarsh, Theory of Functions, § 9.51.
\(\ddagger\) Hardy, Littlewood, and Polya, Inequalities, Theorem 58.

7.12. An alternative set of mean-value theorems.\(\ddagger\) Instead of considering integrals of the form
\[
J(T) = \int_{-\delta}^{\delta} (\sigma + it)^{2s} dt
\]
where \( T = T_{\delta} \) is large, we shall now consider integrals of the form
\[
J(\delta) = \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{itT_{\delta}^{2s}} dt
\]
where \( \delta \) is small.

The behaviour of these two integrals is very similar. If \( J(\delta) = O(1/\delta) \), then
\[
I(T) < e^{-T_{\delta}^{2s}} \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{-itT_{\delta}^{2s}} dt < \epsilon J(1/\delta) = O(T).
\]
Conversely, if \( I(T) = O(T) \), then
\[
J(\delta) = \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{itT_{\delta}^{2s}} dt = O(1/\delta).
\]

Similar results plainly hold with other powers of \( T \), and with other functions, such as powers of \( T \) multiplied by powers of \( \log T \).

We have also more precise results; for example, if \( I(T) \sim CT \), then \( J(\delta) \sim C/\delta \), and conversely.

If \( I(T) \sim CT \), then \( J(\delta) \sim C/\delta \) and conversely.

\[
J(\delta) = \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{itT_{\delta}^{2s}} dt = \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{itT_{\delta}^{2s}} dt = \int_{-\delta}^{\delta} (\sigma + it)^{2s} e^{itT_{\delta}^{2s}} dt.
\]

The last term is \( C \delta^{-2s}(T_{\delta} + 1/5) \), and the modulus of the previous term does not exceed \( C \delta^{-2s}(T_{\delta} + 1/5) \). That \( J(\delta) \sim C/\delta \) plainly follows on choosing first \( T_{\delta} \) and then \( \delta \).

The converse follows in the analogue for integrals of the well-known Tauberian theorem of Hardy and Littlewood,\(\ddagger\) viz. that if \( \alpha \delta \to 0 \), and
\[
\sum_{n \geq 1} \alpha_n n^s \sim \frac{1}{1 - s}(2 \to 1)
\]
then
\[
\sum_{n \geq 1} \alpha_n \sim N.
\]

\(\ddagger\) See Titchmarsh, Theory of Functions, §§ 7.31-7.33.
The theorem for integrals is as follows:

If $f(t) > 0$ for all $t$, and

$$
\int_0^\infty f(t)e^{-xt}dt \sim \frac{1}{x^\delta}
$$

(7.12.1)

as $\delta \to 0$, then

$$
\int_0^\infty f(t) dt \sim T
$$

(7.12.2)

as $T \to \infty$.

We first show that, if $P(x)$ is any polynomial,

$$
\int_0^\infty f(t)e^{-xt}P(e^{-x})dt \sim \frac{1}{x} \frac{1}{\delta} P(x) dx.
$$

It is sufficient to prove this for $P(x) = x^p$. In this case the left-hand side is

$$
\int_0^\infty f(t)e^{-xt} t^p dt \sim \frac{1}{(k+1)\delta} \frac{1}{\delta} \int_0^\infty x^p dx.
$$

Next, we deduce that

$$
\int_0^\infty f(t)e^{-xt} g(e^{-x})dt \sim \frac{1}{x} \frac{1}{\delta} g(x) dx
$$

(7.12.3)

if $g(x)$ is continuous, or has a discontinuity of the first kind. For, given $\epsilon$, we can construct polynomials $p(x)$, $P(x)$, such that

$$
p(x) \leq g(x) \leq P(x)
$$

Then

$$
\lim_{\delta \to 0} \int_0^\infty f(t)e^{-xt} g(e^{-x})dt \leq \lim_{\delta \to 0} \int_0^\infty f(t)e^{-xt} P(e^{-x})dt
$$

and making $\epsilon \to 0$ we obtain

$$
\lim_{\delta \to 0} \int_0^\infty f(t)e^{-xt} g(e^{-x})dt \leq \int_0^\infty g(x) dx.
$$

Similarly, arguing with $p(x)$, we obtain

$$
\lim_{\delta \to 0} \int_0^\infty f(t)e^{-xt} g(e^{-x})dt \geq \int_0^\infty g(x) dx,
$$

and (7.12.3) follows.

† See Titchmarsh, Theory of Functions, §7.53.

Now let

$$
g(x) = 0 \quad (0 \leq x < e^{-1}), \quad = 1/x \quad (e^{-1} \leq x \leq 1).
$$

Then

$$
\int_0^\infty f(t)e^{-xt} g(e^{-x}) dt \sim \frac{1}{x} \int_0^1 f(t) dt
$$

and

$$
\int_0^\infty g(x) dx \sim \frac{1}{x} \int_0^1 dx = 1.
$$

Hence

$$
\int_0^\infty f(t) dt \sim \frac{1}{\delta},
$$

which is equivalent to (7.12.2).

If $f(t) \geq 0$ for all $t$, and, for a given positive $m$,

$$
\int_0^\infty f(t)e^{-xt} dt \sim \frac{1}{x} \frac{1}{\delta} \log^m \frac{1}{\delta}
$$

(7.12.4)

then

$$
\int_0^\infty f(t) dt \sim T \log^m T.
$$

(7.12.5)

The proof is substantially the same. We have

$$
\int_0^\infty f(t)e^{-xt} dt \sim \frac{1}{(x+1)^\beta} \log^m \frac{1}{(x+1)^\beta} \sim \frac{1}{(x+1)^\beta} \log^m \frac{1}{\delta}
$$

and the argument runs as before, with $\frac{1}{x}$ replaced by $\frac{1}{x} \log^m \frac{1}{\delta}$.

We shall also use the following theorem:

If

$$
\int_0^1 f(t)e^{-xt} dt \sim \beta^{-n} \quad (n > 0),
$$

(7.12.6)

then

$$
\int_0^1 f(t)e^{-xt} dt \sim \frac{1}{p(\alpha)} \beta^{-n} \quad (0 < \beta < n).
$$

(7.12.7)

Multiplying (7.12.8) by $(\gamma - \eta)^{\beta - 1}$ and integrating over $(\eta, \infty)$, we obtain

$$
\int_0^1 f(t) dt \int_0^{\infty} e^{-t(\gamma - \eta)^{\beta - 1} \delta} d\delta = C \int_0^1 (\delta - \eta)^{\beta - 1} \delta d\delta,
$$

and
Now
\[
\int \frac{e^{-\eta(\beta - \eta)\gamma}}{\eta} \, d\eta = e^x \int_0^\infty \frac{e^{-\eta x \beta}}{\eta} \, d\eta = e^x \eta^{-1} \Gamma(\beta, \eta),
\]
\[
\int \frac{\eta^{-1} \Gamma(\beta, \eta)}{\gamma} \, d\gamma = \int_0^\infty \frac{\eta^{-1} \Gamma(\beta, \eta)}{\gamma} \, d\gamma = \frac{\Gamma(\beta)}{\beta} \Gamma(\beta - 1, \eta),
\]
and the remaining term in plainly \(o(\eta^s)\) as \(\eta \to 0\). Hence the result.

**7.13.** We can approximate to integrals of the form \(J(\beta)\) by means of Parseval's formula. If \(R(x) > 0\), we have
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(s)J(s) \, ds = \sum_{n=1}^{\infty} \frac{d_n(s)}{2\pi i} \int_{-\infty}^{\infty} \Gamma(s)J(s) \, ds = \sum_{n=1}^{\infty} d_n(s) \Gamma(s),
\]
the inversion being justified by absolute convergence. Now move the contour to \(s = 0\) (\(0 < s < 1\)). Let \(R_0(s)\) be the residue at \(s = 1\), so that \(R_0(s)\) is of the form
\[
\frac{1}{\Gamma(\beta)} \left( d(1+\alpha_1) \log \gamma + \cdots + d(1+\alpha_k) \log \gamma^{\alpha_k} \right).
\]
Let
\[
\phi_1(s) = \sum_{n=1}^{\infty} d_n(s) \Gamma(s),
\]
Then
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(s)J(s) \, ds = \phi_1(s).
\]
Putting \(z = \text{inc}^{\beta}\), where \(0 < \beta < 1\), we see that
\[
\phi_1(\text{inc}^{\beta}),
\]
are Mellin transforms. Hence the Parseval formula gives
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\phi_1(\text{inc}^{\beta})} (\text{inc}^{\beta})^{\alpha_k} \, ds = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(s)J(s)}{\phi_1(\text{inc}^{\beta})} (\text{inc}^{\beta})^{\alpha_k} \, ds.
\]
Now as \(\alpha_k \to \infty\)
\[
\Gamma(s)J(s) = e^{-\text{inc}^{\beta}} \Gamma(s)J(s) = \frac{1}{\text{inc}^{\beta(1+O(\text{inc}^{-1}))}}
\]
Hence the part of the \(t\)-integral over \((\infty, 0)\) is bounded as \(s \to 0\), and we obtain, for \(\frac{1}{2} < s < 1\),
\[
\int_0^1 \frac{\text{inc}^{\beta-1}(1+O(\text{inc}^{-1})) \Gamma(\beta) \gamma^{-1} \, d\gamma = \int_0^1 \text{inc}^{\beta-1}(1+O(\text{inc}^{-1})) \gamma^{-1} \, d\gamma = O(1).
\]
(7.13.4)

**7.13.** **Mean-value Theorems**

In the case \(s = \frac{1}{2}\), we have
\[
\frac{1}{\Gamma(\beta + i\gamma)} = \pi \text{coth} \pi \gamma - 2 \pi \text{coth} \pi \gamma + O(\pi \gamma),
\]
The integral over \((\infty, 0)\), and the contribution of the \(O\)-term to the whole integral, are now bounded, and in fact are analytic functions of \(\delta\), regular for sufficiently small \(\delta\). Hence we have
\[
\int_0^\infty \frac{1}{\Gamma(\beta + i\gamma)} \, d\gamma = \int_0^\infty \frac{1}{\text{inc}^{\beta-i\gamma}} \, d\gamma = O(1).
\]
(7.13.5)

**7.14.** We now apply the above formulas to prove

**Theorem 7.14.** \(A \delta \to 0\)

\[
\int_0^\infty \frac{1}{\text{inc}^{\beta}} \, d\gamma = \frac{1}{\beta} \log \frac{1}{\delta}.
\]
In this case \(R_0(s) = 1/\pi\), and
\[
\phi_1(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(s)}{\phi_1(\text{inc}^{\beta})} (\text{inc}^{\beta})^{\alpha_k} \, ds = O(1).
\]
Hence (7.13.5) gives
\[
\int_0^\infty \frac{1}{\text{inc}^{\beta}} \, d\gamma = \int_0^\infty \frac{1}{\text{inc}^{\beta}} \, d\gamma = \frac{1}{\beta} \log \frac{1}{\delta}.
\]
(7.14.1)

The \(\beta\)-integral is bounded uniformly in \(\delta\) over \((0, \pi)\), so that this part of the integral is \(O(1)\). The remainder is
\[
\int_\pi^\infty \left[ \frac{1}{\text{inc}^{\beta}} \right] \, d\gamma = \int_\pi^\infty \frac{1}{\exp(\text{inc}^{-\beta}) - 1} \frac{1}{\exp(-\text{inc}^{-\beta}) - 1} \, d\gamma
\]
\[
= \int_\pi^\infty \frac{1}{\exp(\text{inc}^{-\beta}) - 1} \frac{\text{inc}^{\beta} \, dx}{\exp(-\text{inc}^{-\beta}) - 1} = -\text{inc}^{\beta} \int_\pi^\infty \frac{1}{\exp(-\text{inc}^{-\beta}) - 1} \, d\gamma
\]
\[
- \text{inc}^{\beta} \int_\pi^\infty \frac{1}{\exp(-\text{inc}^{-\beta}) - 1} \, d\gamma = -\text{inc}^{\beta} \int_\pi^\infty \frac{1}{\exp(-\text{inc}^{-\beta}) - 1} \, d\gamma
\]
(7.14.2)

The last term is a constant. In the second term, turn the line of integration round to \((\pi, -\pi)\). The integral is then regular on the contour for sufficiently small \(\delta\), and is \(O(\pi \exp(-\pi \delta))\) as \(\delta \to \infty\). This integral is therefore bounded; and similarly so is the third term.
The first term is
\[
\frac{1}{\sqrt{2\pi}} \sum_{m,n=1}^{\infty} e^{-\frac{1}{2}(m+n+1)^2} = \frac{1}{\sqrt{2\pi}} \sum_{m,n=1}^{\infty} e^{-\frac{1}{2}(m+n+1)^2}
\]

and the terms with \( m \neq n \) give
\[
O\left(\sum_{m,n=1}^{\infty} e^{-\frac{1}{2}(m+n+1)^2} \right) = O\left(\frac{1}{\sqrt{2\pi}} \log \frac{1}{\sqrt{2\pi}} \right).
\]

Hence
\[
\int_{\sigma}^{\sigma+\delta} \left(\Gamma(\sigma+i) \Gamma(\sigma+i)\right) dt \sim \frac{\Gamma(\sigma+i) \Gamma(\sigma+i)}{2\pi i \sigma}
\]

Hence by (7.12.6), (7.12.7)
\[
\int_{\sigma}^{\sigma+\delta} \left(\Gamma(\sigma+i) \Gamma(\sigma+i)\right) dt \sim \frac{\Gamma(\sigma)}{2\sigma}
\]

7.15. We shall now show that we can approximate to the integral (7.14.1) by an asymptotic series in positive powers of \( \delta \).

We first require

THEOREM 7.15. As \( \sigma \to 0 \) in any angle \( |\arg z| \leq \lambda \), where \( \lambda < \frac{\pi}{4} \),
\[
\sum_{n=1}^{\infty} \frac{a_n e^{-nz}}{n^s} \sim \frac{2}{\pi^2} \sum_{n=1}^{\infty} b_n e^{-\Re s} + O\left(\frac{e^{-\Re s}}{\sqrt{s}}\right),
\]

where the \( b_n \) are constant.

Near \( s = 1 \)
\[
\Gamma(1) \Gamma(\sigma s) = (1-\gamma)(1-\gamma) \cdots \left(1 - \frac{1}{1} + \gamma \cdots \right)^{\sigma} (1-\sigma \log z + \ldots)
\]

Hence by (7.13.1), with \( \lambda = 2 \),
\[
\sum_{n=1}^{\infty} \frac{a_n e^{-\sigma n}}{n^s} \sim \frac{2}{\pi^2} \sum_{n=1}^{\infty} b_n e^{-\sigma n} + O\left(\frac{e^{-\sigma n}}{\sqrt{s}}\right)
\]

The constant implied in the O, of course, depends on \( N \), and the series taken to infinity is divergent, since the function \( \sum_{n=1}^{\infty} a_n e^{-\sigma n} \) cannot be continued analytically across the imaginary axis.

\( \Gamma \) Wimp (1)
We can now prove the

**Theorem 7.15 (A).** As \( s \to 0 \), for every positive \( N \),
\[
\int_{\frac{1}{2}}^{1} z \sum_{n \leq x} \frac{1}{(n+1)^{1+it}} dt = \frac{\gamma - \log 2\pi}{2} + \sum_{n \geq 2} c_n b_n + O\left(e^{\delta x^{1/2}}\right)
\]
the constant of the \( O \) depending on \( N \), and the \( c_n \) being constants.

We observe that the term \( O(1) \) in (7.14.2) is
\[
\frac{1}{z} \sum_{n \leq x} \frac{1}{(n+1)^{1+it}} \left[ \zeta(1+it) - \frac{1}{2} \right] dt - \frac{1}{z} \sum_{n \leq x} \frac{1}{(n+1)^{1+it}} \left[ \zeta(1+it) - \frac{1}{2} \right] dt,
\]
and is thus an entire function of \( \delta \), regular for \( |\delta| < \pi \), also
\[
\int_{\frac{1}{2}}^{1} \left( \frac{1}{\exp(iz\delta)} - 1 \right) \left( \frac{1}{\exp(-iz\delta)} - 1 \right) \frac{1}{z} \left( \exp(-iz\delta) - 1 \right) \frac{1}{\exp(-iz\delta) - 1} \frac{1}{z} \left( \exp(-iz\delta) - 1 \right) \frac{1}{\exp(-iz\delta) - 1} \frac{dx}{x}
\]
is analytic for sufficiently small \( |\delta| \). We dissect the remainder of the integral on the right of (7.14.3) as in (7.14.3). As before
\[
\int_{\frac{1}{2}}^{1} \left( \frac{1}{\exp(-iz\delta)} - 1 \right) \frac{dx}{x} = \int_{\frac{1}{2}}^{1} \frac{1}{\exp(-iz\delta)} - 1 \frac{dx}{x}
\]
and the integrand is regular on the new line of integration for sufficiently small \( |\delta| \), and, if \( \delta = \xi + iy \), this is \( O(y^{-1} \exp(-y \cos \xi \pi)) \) as \( y \to \infty \). The integral is therefore regular for sufficiently small \( \delta \). Similarly, for the third term on the right of (7.14.3) and the fourth term is a constant.

By the calculus of residues, the first term is equal to
\[
2\pi i \sum_{n \leq x} \frac{1}{\exp(-2\pi n^2 \text{rad}) - 1} \int_{C} \frac{1}{\exp(-2\pi n^2 \text{rad}) - 1} \frac{dy}{y}
\]
As before, the \( y \)-integral is an analytic function of \( \delta \), regular for \( |\delta| < \pi \) small enough. Expressing the series as a power series in \( \exp(2\pi n^2 \text{rad}) \), we therefore obtain
\[
\frac{1}{z} \sum_{n \leq x} \frac{1}{(n+1)^{1+it}} \left[ \zeta(1+it) - \frac{1}{2} \right] dt - \frac{1}{z} \sum_{n \leq x} \frac{1}{(n+1)^{1+it}} \left[ \zeta(1+it) - \frac{1}{2} \right] dt
\]
for \( |\delta| < \pi \) small enough and \( R_1 > 0 \).

† Kobor [4], Aitken [3].
We also use the results
\[ \sum_{\eta = 1}^{\infty} \eta^k e^{-\eta^p} = O\left( \frac{\log \eta}{\eta} \right) \]  \hspace{1cm} (7.16.3) \\
\[ \sum_{\eta = 1}^{\infty} \eta^k e^{-\eta^p} = O\left( \frac{\log^k \eta}{\eta} \right) \]  \hspace{1cm} (7.16.4)

as \( \eta \to 0 \). By (1.2.10)
\[ \frac{1}{2x^4} \int_{2x}^{\infty} \Gamma(x) \frac{\Gamma(x)}{\Gamma(2x)} \eta^{-p} \, dx = \frac{1}{2x^4} \sum_{\eta = 1}^{\infty} \eta^k e^{-\eta^p} \frac{\Gamma(x)}{\Gamma(2x)} \eta^{-p} \, dx \\
= \frac{1}{2x^4} \sum_{\eta = 1}^{\infty} \eta^k e^{-\eta^p} \]  \hspace{1cm} (7.16.5)

Hence
\[ \sum_{\eta = 1}^{\infty} \eta^k e^{-\eta^p} = R + \frac{1}{2x^4} \int_{2x}^{\infty} \Gamma(x) \frac{\Gamma(x)}{\Gamma(2x)} \eta^{-p} \, dx \]  \hspace{1cm} (\frac{1}{2} < \frac{1}{\eta} < 1) \\
= R + O(\eta^{-p}),
\]
where \( R \) is the residue at \( s = 1 \) and
\[ R = \frac{1}{12} \left( \frac{\log \eta}{\eta} \right) + \frac{1}{12} \left( \frac{\log^2 \eta}{\eta} \right) \frac{\log \eta}{\eta}. \]
where \( a, b, c, d \) are constants, and in fact
\[ a = \frac{1}{3(2)} = \frac{1}{3}. \]
This proves (7.16.3); and (7.16.4) can be proved similarly by first differentiating (7.16.5) twice with respect to \( \eta \).
We can now prove†

**Theorem 7.16.** As \( \delta \to 0 \)
\[ \int_{\delta}^{x} \left( \frac{\log \eta}{\eta} \right) e^{-\eta^p} \, d\eta \sim \frac{1}{\eta^p \log \eta}. \]

Using (7.13.5), we have
\[ \int_{\delta}^{x} \left( \frac{\log \eta}{\eta} \right) e^{-\eta^p} \, d\eta = \int_{\delta}^{x} \left( \frac{\log \eta}{\eta} \right) e^{-\eta^p} \, d\eta + O(1), \]
and it is sufficient to prove that
\[ \int_{\delta}^{x} \left( \frac{\log \eta}{\eta} \right) e^{-\eta^p} \, d\eta \sim \frac{1}{\eta^p \log \eta}. \]
† Thmamath (1).

For then, by (7.16.2),
\[ \int_{\delta}^{x} \left( \phi_h(\alpha \eta^p) \right) e^{-\eta^p} \, d\eta = \int_{\delta}^{x} \left( \phi_h(\alpha \eta^p) \right) e^{-\eta^p} \, d\eta = \int_{\delta}^{x} \left( \phi_h\left( \frac{1}{\alpha \eta^p} \right) \right) e^{-\eta^p} \, d\eta \]
\[ = \int_{\delta}^{x} \left( 2 \eta^p \phi_h\left( \frac{1}{\alpha \eta^p} \right) + O(\eta^{-1}) \right) e^{-\eta^p} \, d\eta \]
\[ = \int_{\delta}^{x} \left( \phi_h(\alpha \eta^p) \right) e^{-\eta^p} \, d\eta + O\left( \int_{\delta}^{x} \left( \phi_h(\alpha \eta^p) \right) e^{-\eta^p} \, d\eta \right) \]
\[ = \int_{\delta}^{x} \left( \phi_h(\alpha \eta^p) \right) e^{-\eta^p} \, d\eta + O\left( \frac{1}{\eta^p \log \eta} \right) + O(1), \]
and the result follows.
It is then sufficient to prove that
\[ \int_{\delta}^{x} \left( \sum_{\eta = 1}^{\infty} \log \eta \right) e^{-\eta^p} \, d\eta \sim \frac{1}{\eta^p \log \eta}. \]
for the remainder of (7.16.1) will then contribute \( O(\delta^{-1} \log \delta) \).
As in the previous proof, the left-hand side is equal to
\[ \sum_{\eta = 1}^{\infty} \frac{\log \eta}{\eta} e^{-\eta^p} \delta \]
\[ + \frac{2}{2\sin \delta} \sum_{\eta = 1}^{\infty} \frac{\log \eta}{\eta} (m-n) \sin \delta \cos (2(m-n)\pi \cos \delta) \]
\[ - \frac{\delta}{2\sin \delta} \sum_{\eta = 1}^{\infty} \frac{\log \eta}{\eta} (m-n) \sin \delta \cos (2(m-n)\pi \cos \delta). \]
Now
\[ \Sigma_1 = \frac{1}{2\sin \delta} \left( 1 - e^{-\delta \sin \delta} \right) \sum_{\eta = 1}^{\infty} \frac{\log \eta}{\eta} \delta \]
\[ \sim \frac{1}{2\sin \delta} \sum_{\eta = 1}^{\infty} \frac{\log \eta}{\eta} \delta \sin \delta \int_{\delta}^{x} \frac{\log \eta}{\eta} \delta \sin \delta \, d\eta \]
\[ = \frac{1}{2\sin \delta} \int_{\delta}^{x} \frac{\log \eta}{\eta} \delta \sin \delta \, d\eta \sim \frac{1}{2\sin \delta} \log \delta. \]
\[ |\Sigma_2| \leq 2 \sum_{m=1}^{\infty} \frac{2m \sin \delta}{m \eta} \sum_{n=1}^{m-1} \frac{d(m)d(n)}{(m \eta)^{2n-2}} e^{-\text{Im} \text{sin} \delta} \]
\[ = 4 \sin \delta \sum_{m=1}^{\infty} m \frac{d(m)}{m \eta} e^{-\text{Im} \text{sin} \delta} \sum_{n=1}^{m-1} \frac{d(m-n)}{n^2} \]
\[ = 4 \sin \delta \sum_{m=1}^{\infty} \frac{1}{m \eta} \sum_{n=1}^{m-1} m \frac{d(m)d(n)e^{-\text{Im} \text{sin} \delta}}{n^2} \]

The square of the inner sum does not exceed
\[ \sum_{m=1}^{\infty} \frac{m^2 \eta^2}{m^2} e^{-2\text{Im} \text{sin} \delta} \sum_{n=1}^{m-1} \frac{d(m-n)}{n^2} \]
\[ \leq \sum_{m=1}^{\infty} \frac{m^2 \eta^2}{m^2} e^{-2\text{Im} \text{sin} \delta} \sum_{n=1}^{m-1} \frac{d(m-n)}{n^2} \]
\[ = O(1/\log^2 1/\delta) \] by (7.16.3) and (7.16.4). Hence
\[ \Sigma_2 = O(1/\log^2 1/\delta) \]

Finally (as in the previous proof)
\[ \Sigma_4 = O(\delta^{1/2}(\log 1/\delta)^{1/2}) \]

This proves the theorem.

It has been proved by Atkinson [3] that
\[ \int_{-\infty}^{\infty} \left( \frac{1}{\log^2 1/\delta} \right) e^{-\log 1/\delta} \] 
\[ = \frac{1}{\pi} \int_0^\infty e^{-x} \log^2 x \, dx = 2 \log 2 + B \log^2 2 + B \log 2 + B = O\left(\frac{\log^2 1/\delta}{\pi^2}\right), \]
where
\[ A = \frac{1}{2\pi}, \quad B = -\frac{1}{\pi^2} \left(2 \log 2 - 8 \gamma + 24 \zeta(2)/\pi^2\right) \]

A method is also indicated by which the index [12] could be reduced to 1.

### 7.17

The method of residues used in §7.15 for \([1/4+i]^{\alpha}\) suggests another method of dealing with \([1/4+i]^{\alpha}\). This is primarily a question of approximating to
\[ \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} d(n)e^{-\text{Im} \text{sin} \theta} \, dx = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(1+\text{Im} \text{sin} \theta)^2} \, dx \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} \frac{e^{\text{Im}(\text{sin} \theta) \cdot n}}{\text{Im}(\text{sin} \theta)^2} \, dx \]

In the terms with \(n \geq m\), put \(z = \xi/m\). We get
\[ \sum_{n=1}^{m} \frac{1}{m} \sum_{\xi=1}^{m} \int_{-\infty}^{\infty} \frac{e^{\text{Im}(\text{sin} \theta) \cdot \xi}}{(1+\text{Im} \text{sin} \theta)^2} \, dz \]

Approximating to the integral by a sum obtained from the residues of the first factor, as in §7.15, we obtain an approximation to this
\[ \sum_{n=1}^{m} \frac{1}{m} \sum_{\xi=1}^{m} \int_{-\infty}^{\infty} \frac{e^{\text{Im}(\text{sin} \theta) \cdot \xi}}{(1+\text{Im} \text{sin} \theta)^2} \, dz \]
\[ = 2e^{\text{Im}(\text{sin} \theta) \cdot m} \int_{-\infty}^{\infty} \frac{e^{\text{Im}(\text{sin} \theta) \cdot \xi}}{(1+\text{Im} \text{sin} \theta)^2} \, dz \]
\[ = 2e^{\text{Im}(\text{sin} \theta) \cdot m} \sum_{\xi=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} \frac{e^{\text{Im}(\text{sin} \theta) \cdot \xi}}{(1+\text{Im} \text{sin} \theta)^2} \, dz \]
\[ = O\left(\frac{\log 1/\delta}{\sqrt{\pi}}\right) \]

The terms with \(m < \xi < m\) are
\[ O\left(\frac{\log 1/\delta}{\sqrt{\pi}}\right) \]

The remaining terms are
\[ O\left(\frac{\log 1/\delta}{\sqrt{\pi}}\right) \]
Using Schwärz's inequality and (7.15.3) we obtain

\[ O\left( \sum_{n=1}^{nx} \frac{a_n}{n} \log^\frac{1}{8} n \right) = O\left( \frac{1}{8} \log^\frac{1}{8} \right). \]

Actually it follows from a theorem of Ingham (1) that this term is

\[ O\left( \frac{1}{8} \log^\frac{1}{8} \right). \]

7.18. There are formulae similar to those of § 7.16 for larger values of \( k \), though in the higher cases they fail to give the desired mean-value formula.†

We have

\[ \phi_k^{(2)}(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(s+it)^{-1} \hat{a}(t) \, dt \]

\[ = \frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} \Gamma(1-s)\theta(1-s)t^{-s} \hat{a}(t) \, dt \]

\[ = \frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} \Gamma(1-s)\theta(1-s)t^{-s} \hat{a}(t) \, dt \]

\[ \Gamma(1-s)\theta(1-s) = \left( \frac{\pi}{\sin \pi s} \right)^{1/2} \Gamma(1-s). \]

Now \( \Gamma(1-s)\theta(1-s) = \left( \frac{\pi}{\sin \pi s} \right)^{1/2} \Gamma(1-s) \).

For large \( s \)

\[ \Gamma(1-s)\theta(1-s) \sim \frac{\pi}{\sin \pi s} \Gamma(1-s). \]

Now \( \Gamma(1-s)\theta(1-s) \sim \frac{\pi}{\sin \pi s} \Gamma(1-s) \).

Hence we may expect to be able to replace \( \Gamma(1-s) \) by \( (k-1)^{\frac{1}{2}+1} \frac{\pi}{\sin \pi s} \Gamma(1-s) \).

Also, in the upper half-plane,

\[ \cos \theta \sim \cos \pi s \sim \sim \frac{\pi}{2\pi i} \Gamma(1-s). \]

We should thus replace \( \Gamma(1-s) \) by

\[ \sim \pi \Gamma(1-s). \]

Hence this approximation should be

\[ \phi_k^{(2)}(s) = \frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} \Gamma((k-1)s-\frac{1}{2}+1)\hat{a}(t) \, dt. \]

† See also Bullen (3).
Also \[ \int_{\frac{a}{\sqrt{k-1}}}^b \frac{1}{|\sin^{-1}(\frac{x}{b})|} \, dx = \int_{\frac{a}{\sqrt{k-1}}}^b \frac{1}{\left|\sin^{-1}(\frac{x}{b})\right|} \, dx \quad \text{for} \quad a, b > 0. \]

and by the above formula this should be approximately
\[ \frac{2\pi}{k-1} \times \int_{\frac{a}{\sqrt{k-1}}}^b \frac{d\xi}{\sqrt{1-\xi^2}} \cdot \exp\left(-\left(k-1\right)\xi^2\right) \, d\xi. \]

Putting \( x = \xi^2 \), this is
\[ (2\pi) \frac{a}{\sqrt{k-1}} \times \int_{\frac{a}{\sqrt{k-1}}}^b \frac{d\xi}{\sqrt{1-\xi^2}} \cdot \exp\left(-\left(k-1\right)\xi^2\right) \, d\xi, \]

and we can integrate as before. We obtain
\[ K \sum_{n=1}^\infty \frac{d_n(m) d_n(n)}{(m^2+n^2-1)^{3/2}} \times \exp\left(-k^2 \left(\frac{m^2+n^2-1}{m^2+n^2-1}\right) \cdot \sin^2 \left(\frac{k-1}{2}\right)\right). \]

where \( K \) depends on \( k \) only.

The terms with \( m = n \) are
\[ O\left(\frac{1}{m} \cdot \left(\frac{1}{\left(\sin \frac{k}{2}\right)^2}\right)\right). \]

The rest are
\[ O\left(\sum_{m,n} \frac{1}{(m^2+n^2-1)^2} \cdot \exp\left(-ks^2\right)\right). \]

Now
\[ \sum_{m,n} \frac{1}{(m^2+n^2-1)^2} \cdot \exp\left(-ks^2\right) = O\left(\frac{1}{s^2}\right). \]

Hence we obtain
\[ O\left(\sum_{m,n} \frac{1}{(m^2+n^2-1)^2} \cdot \exp\left(-ks^2\right)\right) = O\left(\frac{1}{s^2}\right). \]

Altogether
\[ \int \left[\int \frac{-1}{(k-1)^2} \, dt \right. \int \left. \exp\left(-k^2 \left(\frac{m^2+n^2-1}{m^2+n^2-1}\right) \cdot \sin^2 \left(\frac{k-1}{2}\right)\right) \, dt \right]. \]

and taking \( k \to \infty \), we obtain
\[ \int \left[\int \frac{-1}{(k-1)^2} \, dt \right. \int \left. \exp\left(-k^2 \left(\frac{m^2+n^2-1}{m^2+n^2-1}\right) \cdot \sin^2 \left(\frac{k-1}{2}\right)\right) \, dt \right] \to 0 \quad \text{as} \quad k \to \infty. \]

This is what we should obtain from the approximate functional equation.

7.18. The attempt to obtain a non-trivial upper bound for
\[ \int \left[\int \frac{-1}{(k-1)^2} \, dt \right. \int \left. \exp\left(-k^2 \left(\frac{m^2+n^2-1}{m^2+n^2-1}\right) \cdot \sin^2 \left(\frac{k-1}{2}\right)\right) \, dt \right]. \]

for \( k \to 2 \) fails. But we can obtain a lower bound for it which may be somewhere near the truth; for in this problem we can ignore \( d_n(\sin^{-1}(\theta)) \) for small \( \theta \), since by (7.13.5)
\[ \int \left[\int \frac{-1}{(k-1)^2} \, dt \right. \int \left. \exp\left(-k^2 \left(\frac{m^2+n^2-1}{m^2+n^2-1}\right) \cdot \sin^2 \left(\frac{k-1}{2}\right)\right) \, dt \right] \to 0. \]

and we can approximate to the right-hand side by the method already used.

If \( k \) is any positive integer, and \( \sigma > 1 \),
\[ L(s) = \sum \left(\frac{1}{p}\right)^s = \sum \frac{\left(\frac{1}{p}\right)^s}{p^s} = \sum \frac{d_n(p)}{p^s}. \]

If we replace the coefficient of each term \( p^{-s} \) by its square, the coefficient of each \( n^{-s} \) is replaced by its square. Hence if
\[ F_\sigma(s) = \sum \frac{d_n(p)}{p^s}, \]

then
\[ F_\sigma(s) = \sum \frac{(\left(\frac{1}{p}\right)^s)^2}{p^s} = \sum \frac{d_n(p)}{p^s}. \]

\[ \text{Titchmarsh (4).} \]
say. Thus
\[ f_k \left( \frac{1}{y^k} \right) = 1 + \frac{y^k}{y} + \ldots, \]
and
\[ \left( 1 - \frac{1}{y^k} \right)^{\gamma} f_0 \left( \frac{1}{y^k} \right) = \left( 1 - \frac{y^k}{y^k + \ldots} \right) \left( 1 + \frac{y^k}{y^k + \ldots} \right) = 1 + O \left( \frac{1}{y^{2k}} \right). \]
Hence the product
\[ \prod_{n} \left( 1 - \frac{1}{y^k} \right)^{\gamma} f_0 \left( \frac{1}{y^k} \right) \]
is absolutely convergent for \( \sigma > \frac{1}{2} \), and as \( \sigma \) represents an analytic function, \( g(\sigma) \) say, regular for \( \sigma > \frac{1}{2} \), and bounded in any half-plane \( \sigma > \frac{1}{2} + \delta \); and
\[ F_0 (\sigma) = \Gamma (\sigma) g(\sigma). \]
Now
\[ \sum_{n=1}^{m} \frac{d^n (\sigma)}{d^m (\sigma)} e^{-y m n \sigma} = \frac{1}{2 m i} \int_{C} \Gamma (\sigma) F_0 (\sigma + 2 m n \sin \delta) \, d\sigma. \]
Moving the line of integration just to the left of \( \sigma = 1 \), and evaluating the residue at \( \sigma = 1 \), we obtain in the usual way
\[ \sum_{n=1}^{m} \frac{d^n (\sigma)}{d^m (\sigma)} e^{-y m n \sigma} \sim \frac{C_\delta \log y^{-1}}{\delta} \]
Similarly
\[ \sum_{n=1}^{m} \frac{d^n (\sigma)}{d^m (\sigma)} e^{-y m n \sigma} \sim \frac{1}{2 m i} \int_{C} \Gamma (\sigma) F_0 (\sigma + 1) (2 m n \sin \delta) \, d\sigma \sim C_\delta \log y^{-1} \]
since here there is a pole of order \( k + 1 \) at \( \sigma = 0 \).
We can now prove

**Theorem 7.12.** For any fixed integer \( k \), and \( 0 < \delta \leq 1 \),
\[ \int_{\delta}^{1} \left| \left( \frac{1+i}{1} \right)^{m n} \right| dt \sim \frac{C_\delta \log y^{-1}}{\delta}. \]
The integral on the right of (7.19.1) is equal to (7.18.1) with \( \lambda = 1 \); and
\[ \Sigma_1 \sim \frac{C_\delta \log y^{-1}}{\delta}, \]
while
\[ \Sigma_2 + \Sigma_3 = O \left( \left( \log y^{-1} \right)^{\frac{1}{2}} \right). \]
The result therefore follows.
and Theorem 7.3 follows (with error term \( O(T) \)) on summing over \( 1 \leq t \leq T \). In particular, we see that Theorem 4.11 is sufficient for this purpose, contrary to Titchmarsh's remark at the beginning of §7.3.

We now write
\[
\int_0^T \left( \frac{1}{1 + t} \right) dt = T \log \left( \frac{T}{2\pi} \right) + (2\gamma - 1) T + O(T).
\]

Much further work has been done concerning the error term \( E(T) \). It has been shown by Balasubramanian [1] that \( E(T) < T^{1+\epsilon} \). A different proof was given by Heath-Brown [4]. The estimate may be improved slightly by using exponential sums, and Ivić [9, Corollary 13.5] has sketched the argument leading to the exponent \( \frac{9}{10} + \epsilon \), using a lemma due to Kolesnik [4]. It is no coincidence that this is twice the exponent occurring in Kolesnik's estimate for \( \mu^2 \), since one has the following result.

**Lemma 7.20.** Let \( k \) be a fixed positive integer and let \( t \geq 2 \). Then
\[
(\frac{1}{t} + it)^s < (\log t)^{\max\{1 + \frac{s}{k}, 1 + \frac{s}{2} \}}.
\]

This is a trivial generalization of Lemma 3 of Heath-Brown [2], which is the case \( k = 2 \). It follows that
\[
(\frac{1}{t} + it)^s < (\log t)^{\max\{1 + \frac{s}{k}, 1 + \frac{s}{2} \}}.
\]

Thus, if \( s \) is the infimum of those \( s \) for which \( E(T) < T^s \), then \( \mu^2 \) is given by the method, then is given by (7.20.9).

The connection between estimates for \( (\frac{1}{t} + it) \) and those for \( E(T) \) should not be pushed too far however, for Good [1] has shown that \( E(T) = \Omega(T) \). Indeed Heath-Brown [1] later gave the asymptotic formula
\[
\int_0^T E(t) \, dt = \frac{T}{\pi} \int_0^1 \frac{\phi(t)}{t} \, dt + O(T^{1+\epsilon} \log^2 T)
\]
from which the above \( \Omega \) result is immediate. It is perhaps of interest to

**Note 7.20.** The error term of (7.20.4) must be \( T^{\frac{1}{2}+\epsilon} \), since any estimate \( O[F(T)] \) readily yields \( E(T) \sim (F(T) \log T)^{\frac{1}{2}+\epsilon} \), by an argument analogous to that used in the proof of Lemma 5 in 4.13. It would be nice to reduce the error term in (7.20.4) to \( O(T^{1+\epsilon}) \) so as to include Balasubramanian's bound \( E(T) < T^{1+\epsilon} \).

Higher mean-values of \( E(T) \) have been investigated by Ivić [1] who showed, for example, that
\[
\int_0^T E(t) \, dt \leq T^{1+\epsilon}.
\]

This readily implies the estimate \( E(T) < T^{1+\epsilon} \).

**Mean-value theorems of Heath-Brown and Ivić depend on a remarkable formula for \( E(T) \) due to Atkinson [1]. Let \( 0 < A < B \) be constants and suppose \( AT \), \( N \in A \). Suppose \( N = N(T) = \frac{T}{2\pi} + \frac{NT}{2\pi} + \frac{N^2}{4} \).

Then \( E(T) = 2 + \Xi_1 + \Xi_2 + O(\log^2 T) \), where
\[
\Xi_1 = 2 + \int_0^T \left( \frac{1}{2\pi} \right) \left( \frac{N}{2\pi} + \frac{N^2}{4} \right) \left( \frac{\sinh^{-1} \left( \frac{nT}{2\pi} \right)}{\sqrt{B}} \right) \, dt
\]
with
\[
f(n) = \frac{1}{\pi} + 2T \sinh^{-1} \left( \frac{\pi n}{2\pi} \right) + (\pi n + 2\pi T) \left( \frac{\pi n}{2\pi} \right)
\]
and
\[
\Xi_2 = 2 + \int_0^T \left( \frac{1}{2\pi} \right) \left( \frac{N}{2\pi} + \frac{N^2}{4} \right) \left( \frac{\sinh^{-1} \left( \frac{nT}{2\pi} \right)}{\sqrt{B}} \right) \, dt
\]
where
\[
g(n) = T \log \left( \frac{T}{2\pi n} \right) - T^{-\frac{1}{2}} n.
\]

Atkinson loses a minus sign on (1; p 376). This is corrected above. In applications the above formula one can usually show that \( \Xi_2 \) may be ignored. On the Lindelöf hypothesis, for example, one has
\[
\sum_{n \leq x} d(n) n^{-\frac{1}{2}-\epsilon} = x^\epsilon
\]
for \( x \in T \), so that \( 1_{\mathbb{Z}} \leq T \) by partial summation, and in general one finds
\[
\sum_{n \leq x} \frac{d(n)}{n^s} \text{ for } s > 1.
\]
The sum \( \sum \frac{d(n)}{n^s} \) is clearly analogous to that occurring in the explicit formula (12.4.4) for \( \Lambda(x) \) in Dirichlet’s divisor problem. Indeed, if
\[
(n) = n^{-s} \text{ then the summands of } (7.20.3) \text{ are}
\]
\[
(-1)^n (\frac{2T^{-1}d(n)}{n}) \text{ for } n, (2\pi xT)^{-1} - \frac{\pi}{2} + o\left(\frac{T^{-1}d(n)}{n^2}\right).
\]

7.21. Ingham’s result has been improved by Heath-Brown [4] to give
\[
\int_0^T \frac{d(n)}{n^s} \text{ for } s > 1.
\]
where \( c_s = \frac{2(2\pi)^{-1}}{2} \) and
\[
2 \left( \frac{1}{2} - \frac{1}{2}(2\pi)^{-1} - \frac{1}{2} \right) x^{-2} + \pi^{-2}.
\]
The proof requires an asymptotic formula for
\[
\sum_{n \leq x} d(n) \text{ for } n \text{ and } x.
\]
with a good error term. Uniform in \( x \). Such estimates are obtained in Heath-Brown [4] by applying Weil’s bound for the Kloosterman sum (see §7.24).

7.22. Better estimates for \( c_s \) are now available. In particular we have
\( s \leq \frac{1}{2} \) and \( c_\frac{1}{2} < \delta \). The result on \( c_\frac{1}{2} \) is due to Heath-Brown [8]. To deduce the estimate for \( c_\frac{1}{2} \) one merely uses Gabriel’s convexity theorem (see §3.24), taking \( \pi = \frac{1}{2} \), \( \delta = 2 \), \( \lambda = 1 \), and \( \sigma = \frac{3}{4} \).

The key ingredient required to obtain \( s \leq \frac{1}{2} \) is the estimate
\[
\int_0^T \frac{d(n)}{n^s} \text{ for } n \leq T^2 \text{ (log } T)^{1+s} = \frac{1}{2}
\]
of Heath-Brown [4]. According to (7.20.2) this implies the bound
\[
\mu(\frac{1}{2}) \leq \frac{1}{2}.
\]
In fact, in establishing (7.20.1) it is shown that, if \( d(1 + t) \geq V(0) \) for \( 1 \leq t \leq R \), where \( 0 \leq t \leq T \), then
\[
R \leq T^2 V^{-12}(\log T)^{16}.
\]
and, if \( U \geq T^2 \), then
\[
R \leq T^2 U - 12(\log T)^{16}.
\]
Thus one sees not only that \( c_s \leq \frac{1}{2} \) but also that the number of points at which this bound is close to being attained is very small.

To prove (7.22.1) one uses Atkinson’s formula for \( E(T) \) (see §7.20) to show that
\[
\int_0^T \frac{d(n)}{n^s} \text{ for } n \leq T^2 \text{ (log } T)^{1+s} = \frac{1}{2}
\]
where \( K \) runs over powers of \( 2 \) in the range \( T^{\frac{1}{2}} < K < T^{\frac{1}{2}} \). In order to obtain the estimate (7.22.1) the results of (7.20.7) must be used to estimate how often the sum \( S(x, K, T) \) can be large, for varying \( T \). This is done by using a variant of Hlawka’s method, as described in §6.3.

By following similar ideas, Graham, in work in the process of publication, has obtained
\[
\int_0^T \frac{d(n)}{n^s} \text{ for } n \leq T^2 \text{ (log } T)^{1+s} = \frac{1}{2}
\]
and (7.22.2) contains the estimate \( \mu(\frac{1}{2}) \leq \frac{1}{2} \) (which is the case \( l = \frac{1}{2} \) of Theorem 5.16) in the same way that (7.22.2) implies \( \mu(\frac{1}{2}) \leq \frac{1}{2} \).

7.23. As in §7.9, one may define \( c_\delta \) for all positive real \( k \), as the infimum of those \( c \) for which (7.9.1) holds, and \( c_\delta \) similarly, for (7.9.2).
Then it is still true that $\sigma_\delta = \sigma_\delta'$, and that

$$
\int_0^1 \left| (\omega + it)^{1/2} \right|^2 dt = T \sum_{\alpha} \sigma_\alpha^2 n^{-\frac{1}{2}} + o(T^{1/2})
$$

for $\sigma > \sigma_\delta$, where $\delta = (\alpha, \beta) > 0$ may be explicitly determined. This may be proved by the method of Hasegawa [1]; see also Turganolavie [1]. In particular one may take $\delta(\alpha, \beta) = \frac{1}{2} (\alpha, \beta)$ for $\frac{1}{2} < \sigma < 1$ (with [8: 0.113]) or Turganolavie [1]). For some quite general approaches to these fractional moments the reader should consult Ingham [4] and Bohr and Jessen [6].

Mean values for $e = \frac{1}{2}$ are far more difficult, and in no case other than $k = 1$ or 2 is an asymptotic formula for

$$
\int_0^1 \left| (\omega + it)^{1/2} \right|^2 dt = I_k(T),
$$

say, known, even assuming the Riemann hypothesis. However HeathBrown [7] has shown that

$$
T \left( \log T \right)^{\frac{1}{2}} \ll I_k(T) \ll T \left( \log T \right)^{\frac{1}{2}} \left( k = \frac{1}{2} \right).
$$

Ramachandra [3], [4] having previously dealt with the case $k = \frac{1}{2}$, Jutila [4] observed that the implied constants may be taken to be independent of $k$. We also have

$$
I_k(T) \gg T \left( \log T \right)^{\frac{1}{2}}
$$

for any positive rational $k$. This is due to Ramachandra [4] when $k$ is half an integer, and to Heath-Brown [7] in the remaining cases. (Titchmarsh [1: Theorem 29] states such a result for positive integral $k$, but the reference given there seems to yield only Theorem 7.19, which is weaker.) When $k$ is irrational the best result known is Ramachandra's estimate [5]

$$
I_k(T) \gg T \left( \log T \right)^{\frac{1}{2}} \left( \log \log T \right)^{-\frac{1}{2}}
$$

If one assumes the Riemann hypothesis one can obtain by slightly weaker hypotheses for $I_k(T) \ll T \left( \log T \right)^{\frac{1}{2}}$ and for $I_k(T) \ll T \left( \log T \right)^{\frac{1}{2}}$, $k > 0$.

\[ \text{(7.25.1)} \]
One may now replace $e^{it}$ by $1 + (i/T)$ with negligible error, producing

$$\sum_{n=1}^{m} d(n) \exp \left( \frac{2 \pi i n u}{v} \right) \frac{\exp \left( -\frac{2 \pi i n u}{v} \right)}{v^2} = \frac{1}{2 \pi i} \int_{2-\varepsilon}^{1+\varepsilon} T(s) \frac{\zeta(1-s)}{2\pi i} D(u, v) \frac{ds}{s-1}$$

where

$$D(u, v) = \sum_{n=1}^{m} d(n) \exp \left( \frac{2 \pi i n u}{v} \right) n^{-s}.$$

This Dirichlet series was investigated by Estermann [1], using the function $\zeta(s, a)$ of §2.17. It has an analytic continuation to the whole complex plane, and satisfies the functional equation

$$D(u, v) = 2\pi i \tau \zeta(1-s, a) \left\{ D(1-s, a) + \cos(\pi s) D(1, a) \right\},$$

providing that $(u, v) = 1$. To evaluate our original integral, it is necessary to average over $u$ and $v$, so that one may consider

$$\sum_{(u, v) = 1} \frac{D(1-s, a)}{v} = \sum_{(u, v) = 1} \frac{d(n)n^{-s}}{v} \sum_{\nu=1}^{\infty} \frac{\exp(2\pi i n u)}{\nu}.$$  

for example. In order to get a sharp bound for the innermost sum on the right one introduces the Kloosterman sum:

$$\sum_{n \equiv \nu \mod{w}} \exp \left( \frac{2 \pi i n u}{v} \right) = \sum_{n=1}^{w} \exp \left( \frac{2 \pi i n \nu}{v} \right) \sum_{\nu=1}^{w} \frac{1}{v} \sum_{\nu=1}^{w} \frac{\exp \left( \frac{2 \pi i n (m - \nu \nu)}{v} \right)}{v}.$$

and one can now get a significant saving by using (7.24.2). Notice also that $S(u, v, n, a)$ is averaged over $u$, $v$, and $a$, so that estimates for averages of Kloosterman sums are potentially applicable.

By pursuing such ideas and exploiting the connection with non-holomorphic modular forms, Iwaniec [1] showed that

$$\sum_{n=1}^{m} \left| \int \sum_{r=1}^{\infty} \frac{1}{r} \left( \zeta(1+it) \right)^{it} dt \right| < (R + \Theta \Gamma(\delta - 1)) T^*.$$
VIII

Ω-THEOREMS

8.1. Introduction. The previous chapters have been largely concerned with what we may call Ω-theorems, i.e. results of the form

\[ \Omega(t) = O(\varphi(t)) \]

for certain values of \( \alpha \).

In this chapter we prove a corresponding set of \( \Omega \)-theorems, i.e. results of the form

\[ \Omega(t) = O(\varphi(t)) \]

the \( \Omega \) symbol being defined as the negation of \( \alpha \), so that \( F(t) = O(\varphi(t)) \) means that the inequality \( |F(t)| > A\varphi(t) \) is satisfied for some arbitrarily large values of \( t \).

If, for a given function \( F(t) \), we have both

\[ F(t) = O(\varphi(t)) \]

we may say that the order of \( F(t) \) is determined, and the only remaining question is that of the actual constants involved.

For \( \alpha > 1 \) the problems of \( \Omega(t+\varphi(t)) \) and \( \Omega(t+\varphi(t)) \) are both solved. For \( \frac{1}{3} < \alpha < 1 \) there remains a considerable gap between the \( \Omega \)-results of Chapters V–VI and the \( \Omega \)-results of the present chapter. We shall see later that, on the Riemann hypothesis, it is the \( \Omega \)-results which represent the real truth, and the \( \Omega \)-results which fall short of it. We are always more successful with \( \Omega \)-theorems. This is perhaps not surprising, since an \( \Omega \)-result is a statement about all large values of \( t \), an \( \Omega \)-result about some indefinitely large values only.

8.2. The first \( \Omega \) results were obtained by means of Diophantine approximation, i.e. the approximate solution in integers of given equations. The following two theorems are used.

Dirichlet's Theorem. Given \( N \) real numbers \( a_1, a_2, \ldots, a_N \), a positive integer \( q \), and a positive number \( \varepsilon \), we can find a number \( t \) in the range

\[ \varepsilon \leq r \leq \varepsilon q \]

and integers \( x_1, x_2, \ldots, x_N \), such that

\[ |s_n - a_n x_n| \leq \varepsilon q \]  \( (n = 1, 2, \ldots, N) \).

(8.2.1)

The proof is based on an argument which was introduced and employed extensively by Dirichlet. This argument, in its simplest form, is that, if there are \( m+1 \) points in \( m \) regions, there must be at least one region which contains at least two points.

82

G-THEOREMS

Consider the \( N \)-dimensional unit cube with a vertex at the origin and edges along the coordinate axes. Divide each edge into \( q \) equal parts, and thus the cube into \( q^N \) equal compartments. Consider the \( q^N+1 \) points, in the cube, congruent (mod 1) to the points \( (u_1, u_2, \ldots, u_N) \), where \( u = 0, q, 2q, \ldots, q^Nq \). At least two of these points must lie in the same compartment. If these two points correspond to \( u = u_1, u = u_2, (u_1 < u_2) \), then \( t = u_2 - u_1 \) clearly satisfies the requirements of the theorem.

The theorem may be extended as follows. Suppose that we give \( u \) the values \( 0, 2q, 2q, \ldots, 2q^Nq \). We obtain \( q^N+1 \) points, of which one compartment must contain at least \( m+1 \). Let these points correspond to \( u = u_1, u_2, \ldots, u_m+1 \). Then \( t = u_m - u_{m+1} \) all satisfy the requirements of the theorem.

We conclude that the interval \([q, q^Nq]\) contains at least \( m \) solutions of the inequalities (8.3.2), any two solutions differing by at least \( q \).

8.3. Kronecker's Theorem. Let \( a_1, a_2, \ldots, a_N \) be linearly independent real numbers, i.e. numbers such that there is no linear relation

\[ \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_N a_N = 0 \]

in which the coefficients \( \lambda_n \) are integers not all zero. Let \( b_1, b_2, \ldots, b_N \) be any real numbers, and \( q \) a given positive number. Then we can find a number \( t \) and integers \( x_1, x_2, \ldots, x_N \), such that

\[ |s_n - b_n x_n| \leq \frac{1}{q} \]  \( (n = 1, 2, \ldots, N) \).

(8.3.1)

If all the numbers \( b_n \) are zero, the result is included in Dirichlet's theorem. In the general case, we have to suppose the \( a_n \) linearly independent; for example, if the \( a_n \) are all zero, and the \( b_n \) not all integers, there is in general no \( t \) satisfying (8.3.1). Also the theorem assigns no upper bound for the number \( t \) such as the \( q^N \) of Dirichlet's theorem. This makes a considerable difference to the results which can be deduced from the two theorems.

Many proofs of Kronecker's theorem are known. The following is due to Bohr (15).

We require the following lemma.

Lemma. If \( \phi(z) \) is positive and continuous for \( a < z < b \), then

\[ \lim_{n \to \infty} \left( \int_a^b \phi(z) \, dz \right)^{1/n} = \sup \phi(\alpha) \]

A similar result holds for an integral in any number of dimensions.

† Bohr (13), (16), Bohr and Jessen (3), Steuterman (3), Lettenmeyer (1).
Let \( M = \max \phi(\sigma) \). Then
\[
\begin{aligned}
\left\{ \int \phi(\sigma)^n \, d\sigma \right\} \leq ((b-a)M)^n = (b-a)^n M.
\end{aligned}
\]
Also, given \( c \), there is an interval, \((a, b)\), say, throughout which
\[
\phi(\sigma) \geq M - c.
\]
Hence
\[
\left\{ \int \phi(\sigma)^n \, d\sigma \right\} \geq ((b-a)(M-c))^n = (b-a)^n(M-c),
\]
and the result is clear. A similar proof holds in the general case.

**Proof of Kronecker's theorem.** It is sufficient to prove that we can find a number \( t \) such that each of the numbers
\[
e^{2\pi i (a \sigma - b \sigma)} (n = 1, 2, \ldots, N)
\]
differs from 1 by less than a given \( \epsilon \) or, if
\[
F(t) = 1 + \sum_{n=1}^{N} e^{2\pi i (a \sigma - b \sigma)},
\]
that the upper bound of \(|F(t)|\) for real values of \( t \) is \( N + 1 \). Let us denote this upper bound by \( L \). Clearly \( L \leq N + 1 \).

Let
\[
G(\phi_1, \phi_2, \ldots, \phi_N) = 1 + \sum_{n=1}^{N} e^{2\pi i \phi_n},
\]
where the numbers \( \phi_1, \phi_2, \ldots, \phi_N \) are independent real variables, each lying in the interval \((0, 1)\). Then the upper bound of \(|G|\) is \( N + 1 \), this being the value of \(|G|\) when \( \phi_1 = \phi_2 = \ldots = \phi_N = 0 \).

We consider the polynomial expansions of \(|F(\sigma)|^p\) and \(|G(\phi_1, \ldots, \phi_N)|^p\), where \( p \) is an arbitrary positive integer; and we observe that each of these expansions contains the same number of terms. For, the numbers \( a_1, a_2, \ldots, a_N \) being linearly independent, no two terms in the expansion of \(|F(\sigma)|^p\) fall together. Also the moduli of corresponding terms are equal. Thus if
\[
|G(\phi_1, \phi_2, \ldots, \phi_N)|^p = 1 + \sum_{n=1}^{N} C_n e^{2\pi i \phi_n},
\]
then
\[
|F(\sigma)|^p = 1 + \sum_{n=1}^{N} C_n e^{2\pi i (a \sigma - b \sigma)}.
\]
Say. Now the mean values
\[
F_N = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma)|^p \, d\sigma
\]
and
\[
G_N = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |G(\phi_1, \ldots, \phi_N)|^p \, d\phi_1 \ldots d\phi_N
\]
are equal, each being equal to
\[
1 + \sum C_n^2.
\]
This is easily seen in each case on expressing the squared modulus as a product of conjugates and integrating term by term.

Since \( N + 1 \) is the upper bound of \(|G|\), the lemma gives
\[
\lim_{k \to \infty} G_N^{1/k} = N + 1.
\]
Hence also
\[
\lim_{k \to \infty} F_N^{1/k} = N + 1.
\]
But plainly
\[
F_N \leq L
\]
for all values of \( k \). Hence \( L \geq N + 1 \), and so in fact \( L = N + 1 \). This proves the theorem.

**Theorem 8.8.** If \( \sigma > 1 \), then
\[
|\xi(\sigma)| \leq \zeta(\sigma)
\]
for all values of \( \sigma \), while
\[
|\xi(\sigma)| \geq (1 - \sigma)|\xi(\sigma)|
\]
for some indefinitely large values of \( \sigma \).

We have
\[
|\xi(\sigma)| = \sum_{n=1}^{N} n^{-\sigma} \leq \sum_{n=1}^{N} n^{-\sigma} = \zeta(\sigma),
\]
so that the whole difficulty lies in the second part. To prove this we use Dirichlet's theorem. For all values of \( N \)
\[
\xi(\sigma) = \sum_{a_1 \neq \sigma} \frac{e^{2\pi i a_1 \pi}}{s_{a_1}} + \sum_{a_2 \neq \sigma} n^{-\sigma},
\]
and hence (the modulus of the first sum being not less than its real part)
\[
|\xi(\sigma)| \geq \sum_{n=1}^{N} e^{-\sigma} \cos (\log n) \frac{n^{-\sigma}}{n^{-\sigma}}.
\]
By Dirichlet's theorem there is a number \( t \) \((a_1 \leq a \leq a_2 \pi)\) and integers \( a_1, \ldots, a_N \), such that, for given \( N \) and \( q \) \((q \geq 4)\),
\[
|\log q - a_1 | < \frac{1}{q} (n - 1, 2, \ldots, N).
\]
Hence \( \cos (\log n) \geq \cos (2\pi q) \) for these values of \( n \), and so
\[
\sum_{n=1}^{N} e^{-\sigma} \cos (\log n) \geq \sum_{n=1}^{N} n^{-\sigma} \cos (2\pi q) \sum_{n=1}^{N} n^{-\sigma} > \cos (2\pi q) \zeta(\sigma).
\]
Hence by (8.4.3)

\[ |\xi(a)| \geq \cos(\theta/4)|\xi(\infty)| - \frac{1}{N!} \sum_{n=1}^{N} a^{-n} \]

Now

\[ \xi(a) = \sum_{n=1}^{N} a^{-n} > \int_{1}^{\infty} a^{-n} \, dn = \frac{1}{\log a \cdot 1} \]

and

\[ \sum_{n=1}^{N} a^{-n} < \int_{1}^{\infty} a^{-n} \, dn = \frac{N^{1-\sigma}}{\sigma-1} \]

Hence

\[ |\xi(n)| \geq |\cos(\theta/2) - 2N^{-1} + \xi(n)| \]

and the result follows if \( \theta \) and \( N \) are large enough.

**Theorem 8.4 (A).** The function \( \xi(\sigma) \) is unbounded in the open region \( \sigma > 1, t > 3 > 0 \).

This follows at once from the previous theorem, since the upper bound \( \phi(\sigma) \) of \( \xi(\sigma) \) itself tends to infinity as \( \sigma \to 1 \).

**Theorem 8.4 (B).** The function \( \xi(1+i-t) \) is unbounded as \( t \to \infty \).

This follows from the previous theorem and the theorem of Pólya and Landau. Since \( \xi(1+i-t) \) is bounded, if \( \xi(1+i-t) \) were also bounded \( \xi(s) \) would be bounded throughout the half-plane \( 1 < \sigma < 2, t > 3 \), and this is false, by the previous theorem.

8.5. Dirichlet's theorem also gives the following more precise result:

**Theorem 8.5.** However large \( t_0 \) may be, there are values of \( \sigma \) in the region \( \sigma > 1, t > t_0 \), for which

\[ |\xi(\sigma)| > A \log t \]

Also

\[ |\xi(1+i-t)| > \xi(\log t) \]

Take \( t_0 = 1 \) and \( q = 6 \) in the proof of Theorem 8.4. Then (8.4.4) gives

\[ |\xi(\sigma)| > |\cos(\theta/2) - 2N^{-1} + \xi(\infty)| \]

for \( \theta > 1 \) and \( N \), where we choose \( \xi(\infty) \) to be the integer next above \( \xi(\infty) \). Then

\[ |\xi(\sigma)| > \frac{1}{\log(1+N)} \cdot |\xi(N)| > A \log N \]

for a value of \( \sigma \) such that \( N = A \log t \). The required inequality (8.5.1) then follows from (8.5.4). It remains only to observe that the value of \( t \) in question must be greater than any assigned \( t_0 \), if \( \sigma = 1 \) is sufficiently small; otherwise it would follow from (8.5.3) that \( \xi(s) \) was unbounded.

† Bohr and Landau (1).
Now \( R\left(\sum \frac{1}{p^k}\right) = -\sum \frac{\cos(-\log p_k)}{p_k} \) is a linearly independent. For it follows from the theorem that an integer can be expressed as a product of prime factors in one way only, that there can be no relation of the form
\[
y^a_1y^b_2\cdots y^c_k = 1,
\]
where the \(\lambda_i\)'s are integers, and therefore no relation of the form
\[
\lambda_1\log p_1 + \cdots + \lambda_k\log p_k = 0.
\]

Hence also the numbers \(\log p_k\)'s are linearly independent. It follows therefore from Kronecker's theorem that we can find a number \(\tau\) and integers \(c_1, \ldots, c_k\) such that
\[
\left| \frac{\log \varphi_k}{2\pi} - \tau - x_k \right| \leq \frac{1}{k} (k = 1, 2, \ldots, N).
\]

or
\[
\left| \frac{\log p_k - x - 2\pi x_k}{\pi} \right| \leq \frac{1}{k} (k = 1, 2, \ldots, N).
\]

Hence for these values of \(\tau\)
\[
R\left(\sum \frac{1}{p^k}\right) < \frac{1}{2} \sum \frac{1}{\log p_k} + \sum \frac{1}{n \log p_n}.
\]

Since \(\sum p_k^{-1}\) is divergent, we can, if \(H\) is any assigned positive number, choose \(\varepsilon\) so near to \(1\) that \(\sum p_k^{-\varepsilon} > H\). Having fixed \(\varepsilon\), we can choose \(N = \log p_k > H\). Then
\[
R\left(\sum p^k\right) < -\frac{1}{H} + \frac{1}{H} = 0.
\]

This follows from the previous theorem in the same way as Theorem 8.4 (B) from Theorem 8.4 (A).

We cannot, however, proceed to deduce an analogue of Theorem 8.3 for \(1/(\varepsilon x)\). In proving Theorem 8.3, each of the numbers \(\cos(a \log x)\) has to be made as near as possible to \(1\), and this can be done by Dirichlet's theorem. In Theorem 8.6, each of the numbers \(\cos(\log p_k)\) has to be made as near as possible to \(-1\), and this requires Kronecker's theorem.

Now Theorem 8.5 depends on the fact that we can assign an upper limit to the number \(t\) which satisfies the conditions of Dirichlet's theorem. Since there is no such upper limit in Kronecker's theorem, the corresponding argument for \(1/(\varepsilon x)\) fails. We shall see later that the analogue of Theorem 8.5 is in fact true, but it requires a much more elaborate proof.

8.7. Before proceeding to these deeper theorems, we shall give another method of proving some of the above results.† This method deals directly with integrals of high powers of the functions in question, and so might be described as a short cut, which avoids explicit use of Diophantine approximation.

We write
\[
M(\{f(t)\}) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} f(t+it)^{2} \, dt,
\]

and prove the following lemma.

**Lemma.** Let \(g(x) = \sum x^n a_n\) be absolutely convergent for a given value of \(x\), and let every \(m\) with \(b_m \neq 0\) be prime to every \(n\) with \(c_n \neq 0\). Then for each \(x\)
\[
M(\{g(x)\}) = M(\{g(x)\}) M(\{b(x)\})
\]

By Theorem 7.1
\[
M(\{g(x)\}) = \sum_{n=1}^{\infty} \frac{|b_n|}{n^s} M(\{b(x)\}) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^s}.
\]

Now
\[
g(x) = \sum_{n=1}^{\infty} d_n x^{n-1},
\]

where each \(d_n x^{n-1}\) is the product of two terms \(b_m x^{-s} c_n x^{-s}\). Hence
\[
M(\{g(x)\}) = \sum_{n=1}^{\infty} \frac{|d_n|}{n^s} = \sum_{n=1}^{\infty} \frac{|a_n|}{n^s} M(\{b(x)\}) M(\{\bar{x}\})
\]

We can now prove the analogue for \(1/(\varepsilon x)\) of Theorem 8.4.

**Theorem 8.7.** If \(\varepsilon > 0\), then
\[
\left| \frac{1}{\varepsilon x} - \frac{1}{\varepsilon x} \right| \leq \frac{1}{\varepsilon x}.
\]

for all values of \(x\), while
\[
\left| \frac{1}{\varepsilon x} \right| \leq \frac{1}{\varepsilon x}.
\]

for some indefinitely large values of \(x\).

† Zolf and Landau (7).
We have, for \( \sigma > 1 \),
\[
\left| \frac{1}{\zeta(s)} \right| = \sum_{n=1}^{\infty} \frac{|\zeta(n)|}{n^s} < \sum_{n=1}^{\infty} \frac{|\zeta(n)|}{n^s}.
\]
Since we have also
\[
\sum_{n=1}^{\infty} \frac{|\zeta(n)|}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} \right) = \prod_p \left( 1 - \frac{1}{p^s} \right) = \frac{\zeta(s)}{\zeta(2s)},
\]
and the first part follows.

To prove the second part, write
\[
\frac{1}{\zeta(s)} = \prod_p \left( 1 - \frac{1}{p^s} \right) \zeta_p^s(s),
\]
and by repeated application of the lemma it follows that
\[
M\left( \frac{1}{\zeta(s)}^{|s|} \right) = \prod_p M\left( \left| 1 - \frac{1}{p^s} \right| \right) M\left( \zeta_p^s(s) \right).
\]
Now, for every \( p \),
\[
M\left( \left| 1 - \frac{1}{p^s} \right| \right) = \frac{\log p}{2\pi} \int_{1/2}^{1} \left| 1 - \frac{1}{p^t} \right| dt,
\]
since the integrand is periodic with period \( 2\pi\log p \) and
\[
M\left( \left| \zeta_p^s(s) \right| \right) \geq 1,
\]
since the Dirichlet series for \( \zeta_p^s(s) \) begins with \( 1 + \ldots \). Hence
\[
M\left( \frac{1}{\zeta(s)}^{|s|} \right) \geq \prod_p \frac{\log p}{2\pi} \int_{1/2}^{1} \left| 1 - \frac{1}{p^t} \right| dt.
\]
Now
\[
\lim_{k \to \infty} \left( \int_{1/2}^{1} \left| 1 - \frac{1}{p^t} \right| dt \right)^{1/k} = \max_{\phi : \zeta(s) \neq 0} \left| 1 - \frac{1}{p^t} \right| = 1 + \frac{1}{p^2},
\]
Hence
\[
\lim_{k \to \infty} M\left( \frac{1}{\zeta(s)}^{|s|} \right)^{1/k} \geq \zeta(s) / \zeta(2s).
\]
Since the left-hand side is independent of \( N \), we can make \( N \to \infty \) on the right, and obtain
\[
\lim_{k \to \infty} M\left( \frac{1}{\zeta(s)}^{|s|} \right)^{1/k} \geq \zeta(s) / \zeta(2s).
\]

Hence to any \( \epsilon \) corresponds a \( \delta \) such that
\[
M\left( \frac{1}{\zeta(s)}^{|s|} \right)^{1/2k} > (1 - \epsilon) \frac{\zeta(s)}{\zeta(2s)},
\]
and (8.7.3) now follows.

Since \( \zeta(s) / \zeta(2s) \to 1 \) as \( \sigma \to 1 \), this also gives an alternative proof of Theorem 8.6.

It is easy to see that a similar method can be used to prove Theorem 8.4 (A). It is also possible to prove Theorem 8.4 (B) and 8.6 (A) directly by this method without using the Phragmén–Lindelöf theorem. This, however, requires an extension of the general mean-value theorem for Dirichlet series.

8.5. Theorem 8.8.1. However large \( t \) may be, there are values of \( s \) in the region \( \sigma > 1 \), \( t > q \) for which
\[
\frac{1}{\zeta(1 + i\alpha)} = O(\log \log t).
\]

Also
\[
\frac{1}{\zeta(1 + i\alpha)} = O(\log \log t).
\]

As in the case of Theorem 8.5, it is enough to prove the first part. We first prove some lemmas. The object of these lemmas is to supply, for the particular case in hand, what Krommer’s theorem lacks in the general case, viz. an upper bound for the number \( f \) which satisfies the conditions \((\mathbf{9} \times 1)\).

Lemma 1. If \( m \) and \( n \) are different positive integers,
\[
\log m > \frac{1}{\max(m,n)}.
\]

For if \( m < n \)
\[
\log m > \log n - \log (n - 1) = \frac{1}{(n - 1) + \frac{1}{2} + \cdots + \frac{1}{n}},
\]

Lemma 2. If \( p_1, \ldots, p_r \) are the first \( N \) primes, and \( p_\nu, \ldots, p_r \) are integers, not all 0 (not necessarily positive), then
\[
\log \prod \frac{p_\nu^{\mu}}{n} > p_r^{\mu N} \quad (\mu = \max\{p_\nu\}).
\]

For \( \prod \frac{p_\nu^{\mu}}{n} = u / v \), where
\[
u = \prod p_\nu^{\mu}, \quad v = \prod p_\nu^{\mu}
\]
(Behr and Landau (7)).
and u and v, being mutually prime, are different. Also
\[ \max(u, v) \leq \prod p^e < p_0^{N_0}, \]
and the result follows from Lemma a.

**Lemma 7.** The number of solutions in positive or zero integers of the equation
\[ v_1^k + v_2^k + \cdots + v_N^k = k \]
do not exceed \((k+1)^N\).

For \(N = 1\) the number of solutions is \(k+1\), so that the theorem holds. Suppose that it holds for any given \(N\). Then for given \(v_1, \ldots, v_N\) the number of solutions of
\[ v_1 + v_2 + \cdots + v_N = \sum_{i=1}^N v_i \]
do not exceed \((k-1)^N\); and \(v_1, v_2, \ldots, v_N\) can take \(k-1\) values. Hence the total number of solutions is \(\leq (k+1)^N\), whence the result.

**Lemma 5.** For \(N > A\), there exists a \(t\) satisfying \(0 < t < \exp(N^3)\) for which
\[ \cos(t \log p_n) = -1 + t^2/2 \quad (n < N). \]

Let \(N > 1, k > 1\). Then
\[ \left( \sum_{v_1} v_1 \right)^k = \sum_{v_1, v_2} v_1 v_2 \sum_{v_3} v_3, \]
where \(v_1, v_2, v_3, \ldots \) are distinct integer \(v_i\), \(\sum v_i = k\).

The number of distinct terms in the expansion is at most \((k+1)^N \leq 4^{N^2}\), by Lemma 7. Hence
\[ \left( \sum \right) = \sum \leq 4^{N^2} \sum, \]
so that
\[ \sum \geq k - 4^{N^2} \sum = k - 4^{N^2} (N+1)^2. \]

Let \(F(t) = \frac{1}{N^2} \sum_{v_1} e^{it \log p_n}\), so that
\[ |F(t)|^2 = \sum_{v_1, v_2} e^{-i(\log p_n) \sum} \sum_{v_1, v_2} \left( \sum_{v_1} e^{it \log p_n} \right), \]
\[ |F(t)|^2 = \sum_{v_1} e^{-i(\log p_n) \sum} \left( \sum_{v_1} e^{it \log p_n} \right) = \sum_{v_1} \varphi_p, \]
where \(\varphi_p\) is taken over values of \((v_1, v_2)\) for which \(v_1 = v_2 = \cdots = v_1\),

\[ \sum_\varphi \text{ over the remainder. Now} \]
\[ \frac{1}{N^2} \int_0^\frac{2}\pi e^{it \log p_n} \, dt = 1 \quad (n = 0), \]
\[ \frac{1}{N^2} \int_0^\frac{2}\pi e^{it \log p_n} \, dt = \frac{e^{it \log p_n} - 1}{it \log p_n} \leq \frac{2}{|a|^2} \quad (n \neq 0). \]
Hence
\[ \frac{1}{N^2} \int_0^\frac{2}\pi |F(t)|^2 \, dt \geq \sum_{v_1, v_2} v_1 v_2 \sum \left( \sum_{v_1} \log p_n \right)^2. \]
By Lemma 5, since the numbers \(v_1, v_2\) are not all 0,
\[ \left[ \sum_{v_1, v_2} v_1 \log p_n \right] \geq \sum \left( \sum \log p_n \right)^2 > p_0^{N^3 + \log N - 2} \geq p_0^{N^3}. \]
Hence
\[ \frac{1}{N^2} \int_0^\frac{2}\pi |F(t)|^2 \, dt \geq \sum_{v_1, v_2} v_1 v_2 \sum \left( \sum \log p_n \right)^2 \geq k - 4^{N^2} (N+1)^2. \]
In this we take \(N = A\), \(T = e^{N^3}\), and obtain, for \(N > A\),
\[ \sum_{v_1, v_2} v_1 v_2 \sum \left( \sum \log p_n \right)^2 \geq \sum_{v_1, v_2} \sum \left( \sum \log p_n \right)^2. \]
Hence
\[ \left[ \frac{1}{N^2} \int_0^\frac{2}\pi |F(t)|^2 \, dt \right] \leq \sum_{v_1, v_2} v_1 v_2 \sum \left( \sum \log p_n \right)^2 \geq \sum_{v_1, v_2} \sum \left( \sum \log p_n \right)^2. \]
Hence there is a \(t\) in \((0, e^{N^3})\) such that
\[ |F(t)| > N + 1 - \frac{1}{2N}. \]
Suppose, however, that \(\cos(t \log p_n) = -1 + 1/N\) for some value of \(n\). Then
\[ |F(t)| \leq N + 1 + |1 - e^{it \log p_n}| \leq N + 1 + \sqrt{2} \left( 1 - \frac{1}{N} \right)^{1/2} \leq N + 1 - \frac{1}{2N}, \]
a contradiction. Hence the result.
We can now prove Theorem 8.8. As in § 8.6, for \( \sigma > 1 \)

\[
\log \frac{1}{|\zeta(s)|} = -\sum \frac{\cos\log p_n}{p_n^s} + O(1).
\]

Let now \( N \) be large, \( t = t(N) \) the number of Lemma 6, 3 = 1/log \( N \), and \( \sigma = 1+\delta \). Then

\[
\log \frac{A}{[\zeta(s)]} = \sum \frac{\cos\log p_n}{p_n^s} + \left(1 - \frac{1}{N}\right) \sum \frac{1}{n^s} - \frac{1}{N} \sum \frac{1}{n^s}
\]

\[
> \left(1 - \frac{1}{N}\right) \log \frac{1}{\delta - A} \geq \log \frac{1}{\delta - A} - A \sum \frac{1}{n^s}.
\]

\[
> \log \frac{A}{[\zeta(s)]} > -A \sum \frac{1}{n^s} > \frac{A}{\log N} > 1 > A \log \log t.
\]

The number \( t = t(N) \) evidently tends to infinity with \( N \), since \( 1/|\zeta(s)| \) is bounded in \( |\zeta| < A \), \( \sigma > 1 \), and the proof is completed.

8.9. In Theorems 8.5 and 8.8 we have proved that each of the inequalities

\[
|1/\zeta(s)| > A \log \log t, \quad 1/|\zeta(s)| > A \log \log t
\]

is satisfied for some arbitrarily large values of \( t \), if \( A \) is a suitable constant. We now consider the question how large the constant can be in the two cases.

Since neither \( |1/\zeta(s)|/\log \log t \) nor \( |1/\zeta(s)|^{-1} \log \log t \) is known to be bounded, the question of the constants might not seem to be of much interest. But we shall see later that on the Riemann hypothesis they are both bounded; in fact if

\[
\lambda = \lim_{t \to \infty} \frac{1}{\log \log t}, \quad \mu = \lim_{t \to \infty} \frac{1}{\log \log t},
\]

then, on the Riemann hypothesis,

\[
\lambda \lesssim 2\sigma, \quad \mu \lesssim \frac{12}{\log s},
\]

where \( e \) is Euler's constant.

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There is therefore a certain interest in proving the following results.

**Theorem 8.9 (A).** \( \lim_{t \to \infty} \frac{|1/\zeta(s)|}{\log \log t} \leq \sigma \).

**Theorem 8.9 (B).** \( \lim_{t \to \infty} \frac{|1/\zeta(s)|}{\log \log t} \geq \sigma \).

Thus on the Riemann hypothesis it is only a factor 2 which remains in doubt in each case.

We first prove some identities and inequalities. As in § 7.19, if

\[
F_2(x) = \sum \frac{x_n}{n^s} \quad (\sigma > 1)
\]

and

\[
J_2(x) = \sum \frac{|k|}{(k+1)|m|} \quad (\sigma > 1)
\]

then

\[
F_2(x) = \prod \frac{1}{1 - p^{-s}}.
\]

Now for real \( x \)

\[
J_2(x) = \int \frac{1}{\sqrt{1-x^2}} \sin \left( \frac{x}{2} \right) \frac{dx}{\log x^2}.
\]

Using the familiar formula

\[
P_2(z) = \int \frac{1}{\sqrt{1-z^2}} \sin \left( \frac{z}{2} \right) \frac{dz}{\log z^2}
\]

for the Legendre polynomial of degree \( n \), we see that

\[
J_2(x) = \int \frac{1}{\sqrt{1-x^2}} \sin \left( \frac{x}{2} \right) \frac{dx}{\log x^2}.
\]

Naturally this identity holds also for complex \( x \); it gives

\[
R_2(x) = \prod \frac{1}{1 - p^{-s}} + \prod \frac{1}{1 - p^{-s}}
\]

A similar set of formulas holds for \( S_2(x) \). We have

\[
\frac{1}{|\zeta(s)|^2} = \int \frac{1}{1 - p^{-s}} \prod \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \prod \left( 1 - \frac{1}{p} \right).
\]

\* Littlewood [5], [6], Titchmarsh [4], [14].
Hence
\[ \frac{1}{2\pi i} = \sum_{n=1}^{\infty} b(n) \frac{1}{n^2}, \tag{8.9.10} \]
where the coefficients \( b(n) \) are determined in an obvious way from the above product. They are integers, but are not all positive.

The sum of these coefficients shows that
\[ \sum_{n=1}^{\infty} \frac{b(n)}{n^2} = \prod_{p} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right) = \prod_{p} \left( 1 + \frac{1}{p} \right)^2 = \frac{\zeta(2)}{\pi^2}. \tag{8.9.11} \]
Again, let
\[ G_6(s) = \sum_{n=1}^{\infty} \frac{G(n)}{n^s}. \tag{8.9.12} \]
As in the case of \( F_6(s) \),
\[ G_6(s) = \prod_{p} \left( 1 + \frac{2}{p^s} + \frac{1}{p^{2s}} \right)^{1/2} \prod_{p} \left( 1 + \frac{1}{p^s} \right)^2 = \prod_{p} \zeta(s), \tag{8.9.13} \]
say. Now, for real \( x \),
\[ g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{k^n}{n^s} \sin^{2\pi s} \left( \frac{x}{2\pi} \right) \frac{dk}{k}, \tag{8.9.14} \]
Comparing this with the formula
\[ F_6(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - x^2 \right)^{1/s} \cos x^{1/s} \frac{dx}{x}, \tag{8.9.15} \]
we see that
\[ g(x) = (1 - x^2)^{1/2} F_6 \left( \frac{1 + x}{1 - x} \right). \tag{8.9.16} \]
Hence
\[ G_6(s) = \int_{-\infty}^{\infty} \left( 1 - x^2 \right)^{-s} \cos x^{1/s} \frac{dx}{x}, \tag{8.9.17} \]
We have also the identity
\[ F_6(x) = \left( x^{-1} \right)^{1/2} \log(x). \tag{8.9.18} \]

Again for \( 0 < x < \frac{1}{2} \),
\[ f_6(x) > \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dy}{\left( 1 - 2\sqrt{z} \cos \phi + \frac{z}{2} \right)^2} \]
\[ = \frac{1}{\pi(1 - \sqrt{z})^3} \int_{-\pi}^{\pi} \left( 1 - 2\sqrt{z} \cos \phi + \frac{z}{2} \right)^{-1} d\phi \]
\[ = \frac{1}{\pi(1 - \sqrt{z})^3} \int_{-\pi}^{\pi} \left( 1 + O\left( \frac{1}{z} \right) \right) d\phi \geq \frac{1}{\pi(1 - \sqrt{z})^3} \tag{8.9.19} \]
if \( k \) is large enough. Hence also
\[ g_6(x) = (1 - x)^{2a+1} f_6 \left( \frac{1}{x} \right) > \frac{1}{2k + 2} \left( 1 + \sqrt{2} \right)^{2a+3} \tag{8.9.20} \]
for \( x \) large enough, and
\[ g_6(x) < \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - \sqrt{2} \right)^{2a} d\phi = (1 - \sqrt{2})^{2a} \tag{8.9.21} \]
for all values of \( x \) and \( k \).

8.10. Proof of Theorem 8.9 (A). Let \( \sigma > 1 \). Then
\[ \int_{T_{\pi^2}^n} \left( 1 - \frac{1}{|t|^2} \right)^{1/2} (x + it)^{1/2} \exp \left( \frac{1}{2} \left| \frac{1}{x} \right| \right) \frac{dt}{x} \]
\[ = \sum_{n=1}^{\infty} \frac{d(n)}{n^{\sigma/2}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\sigma/2}} \left( \frac{n}{m} \right)^{\sigma/2} \]
\[ = T \sum_{n=1}^{\infty} \frac{d(n)}{n^{\sigma/2}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\sigma/2}} \left( \frac{n}{m} \right)^{\sigma/2} \]
\[ = T \sum_{n=1}^{\infty} \frac{d(n)}{n^{\sigma/2}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\sigma/2}} \left( \frac{n}{m} \right)^{\sigma/2} \]
\[ = T \sum_{n=1}^{\infty} \frac{d(n)}{n^{\sigma/2}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\sigma/2}} \left( \frac{n}{m} \right)^{\sigma/2} \]
\[ = T \sum_{n=1}^{\infty} \frac{d(n)}{n^{\sigma/2}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\sigma/2}} \left( \frac{n}{m} \right)^{\sigma/2} \]
Since (from its original definition) \( f_6(p^{1/2}) \) is defined for all values of \( p \),
\[ F_6(n \pi / \log x) > \prod_{n=1}^{\infty} f_6(p^{1/2}) > \prod_{n=1}^{\infty} \left( 1 - \frac{1}{p^{1/2}} \right) \tag{8.9.22} \]
for any positive \( x \) and \( k \) large enough. Here the number of factors is
\[ \sum_{n=1}^{\infty} \left( x / \log x \right)^{1/2} \approx \pi(x) \approx x / \log x \approx \pi(x) \approx x / \log x \]
with \( x = \tan^2 \phi \). Hence
\[ \prod_{n=1}^{\infty} \left( 1 - \frac{1}{p^{1/2}} \right) = \exp \left( - \pi(x) / \log x \right) > e^{-x}. \tag{8.9.23} \]
The result now follows by the Phragmén-Lindelöf method. Let

$$f(z) = \frac{\zeta(z)}{\log \log(t+z)}$$

where \( h > 4 \), and let

$$\lambda = \lim \frac{\zeta(1+it)}{\log \log t}.$$

We may suppose \( \lambda \) finite, or there is nothing to prove. On \( \sigma = 1, t > 0 \), we have

$$|f(z)| \leq \frac{|\zeta(z)|}{\log \log t} < \lambda + \epsilon \quad (t > t_0).$$

Also, on \( \sigma = 2, |f(z)| = o(1) < \lambda + \epsilon \quad (t > t_0).$$

We can choose \( \lambda \) so that \( |f(z)| < \lambda + \epsilon \) also on the remainder of the boundary of the strip bounded by \( \sigma = 1, \sigma = 2 \), and \( t = 1 \). Then, by the Phragmén-Lindelöf theorem, \( |f(z)| < \lambda + \epsilon \) in the interior, and so

$$\lim_{T \to \infty} \frac{\zeta(s)}{\log \log t} = \lim_{T \to \infty} \frac{|f(z)|}{\log \log(t+z)} < \lambda.$$  

Hence \( \lambda \geq \sigma \), the required result.

8.11. Proof of Theorem 8.9 (B). The above method depends on the fact that \( d_2(n) \) is positive. Since \( d_2(n) \) is not always positive, a different method is required in this case.

Let \( \sigma > 1 \), and let \( N \) be any positive number. Then

$$\int_{\zeta(n)}^{\infty \zeta(n)} \frac{\zeta(n)}{\log \log x} \, dx = \int_{\zeta(n)}^{\infty \zeta(n)} \frac{\zeta(n)}{\log \log x} \, dx$$

$$\leq \sum_{n=1}^{N} \frac{\zeta(n)}{\log \log x}$$

Hence

$$\zeta(n) \geq 2 \pi \log \log x - 2 \log 2 - 2 \log 3.$$  

Now

$$\zeta(n) \geq \log \log \log x - \log 2 + 1 \geq 1 \geq 2\lambda,$$

so that the last sum does not exceed

$$4N \sum_{n=1}^{\infty} \frac{\zeta(n)}{\log \log \log x} < 4N \sum_{n=1}^{\infty} \frac{\zeta(n)}{\log \log \log x} < 4N \sum_{n=1}^{\infty} \frac{\zeta(n)}{\log \log \log x}.$$  

Since \( \zeta(n) \sim 1/(n-1) \) as \( n \to 1 \), and \( \zeta(n) \to 1 \) as \( n \to 1 \), we have, if \( \sigma \) is sufficiently near to \( 1 \),

$$\frac{\zeta(n)}{\log \log n} < \frac{1}{(n-1).}$$
Hence the above last sum is less than
\[
\frac{4N}{T(\sigma-1)^{2\alpha}}
\]
Also
\[
\frac{1}{N^\theta} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta} \ll \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta} < \frac{1}{N^{1-1}} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta}
\]
\[
< \frac{1}{N^{1-1}} \left( \frac{1}{(\theta+1)^{1/2}} \right) < \frac{1}{N^{1-1}} \left( \frac{2}{(\theta-1)} \right)
\]
for \( \varepsilon \) sufficiently near to 1. Since for \( \sigma > 2 \)
\[
G_2(\varepsilon) < \prod_{n \geq 1} \left( 1 + \frac{1}{n^2} \right)^{1/2} - \prod_{n \geq 1} \left( 1 - e^{-1/n^2} \right)^{1/2} - \left( \frac{1}{\xi(\varepsilon)} \right)^{1/2}
\]
we have similarly
\[
G_2(\varepsilon) = \frac{1}{\xi(\varepsilon)} \sum_{a \in S_N} b_\pm(a) n^{-\theta} < \frac{1}{\xi(\varepsilon)} \sum_{a \in S_N} b_\pm(a) n^{-\theta}
\]
\[
< \frac{1}{\xi(\varepsilon)} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta} < \frac{1}{\xi(\varepsilon)} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta} < \frac{1}{\xi(\varepsilon)} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta}
\]
These two differences are therefore both bounded if
\[
N = \left( \frac{2}{(\theta-1)} \right)^{1/2}
\]
With this value of \( N \) we have
\[
\frac{1}{T} \int \frac{1}{(q(t) + O(1))/\eta} dt = \frac{1}{T} \sum_{a \in S_N} \frac{b_\pm(a)}{n^\theta} dt
\]
\[
> G_2(\varepsilon) - \frac{4N}{T(\sigma-1)^{2\alpha}} + O(1)
\]
\[
> \prod_{n \geq 1} \left( 1 + \frac{1}{n^2} \right)^{1/2} - \frac{4N}{T(\sigma-1)^{2\alpha}} + O(1)
\]
by (8.9.17). Now
\[
\log \prod_{p \leq x} \left( 1 + \frac{1}{p^2} \right) - O(\varepsilon(\sigma-1)\log x)
\]
as in (8.10). Hence, as in (8.10.3) and (8.15.3),
\[
\prod_{p \leq x} \left( 1 + \frac{1}{p^2} \right)^{1/2} > e^{-e^{-t} - t} e^{-\varepsilon(\sigma-1)\log x} |\log x|
\]
where \( b = 6\varepsilon(\sigma-1) \).
8.12. The above theorems are mainly concerned with the neighbourhood of the line \( s = 1 \). We now state one further into the critical strip, and prove it.

**Theorem 8.12.** Let \( \sigma \) be a fixed number in the range \( \frac{1}{2} \leq \sigma < 1 \). Then the inequality

\[
\left| \Gamma(s) \right| > \exp \left( \frac{c}{\log s} \right)
\]

is satisfied for some indefinitely large values of \( s \), provided that

\[ a < 1 - \sigma. \]

Throughout the proof, \( k \) is supposed large enough, and \( \delta \) small enough, for any purpose that may be required. We take \( \frac{1}{2} < \sigma < 1 \), and the constants \( C_1, C_2, \ldots \), and those implied by the symbol \( O \), are independent of \( \delta \) and \( \sigma \), but may depend on \( n \) and on \( x \) when it occurs. The case \( \sigma = \frac{1}{2} \) is deduced finally from the case \( \sigma = \frac{1}{2} - \frac{1}{2} \).

We first prove some lemmas.

**Lemma a.** Let

\[
\Gamma(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(s-1)^n}
\]

in the neighbourhood of \( s = 1 \). Then

\[
|a_n^n| < e^{C_n} \quad (1 \leq m \leq \delta)
\]

The \( a_n^n \) are the same as those of \( \zeta(s) \). We have

\[
\Gamma(s) = \sum_{n=0}^{\infty} C_n (s-1)^n, \quad \zeta(s) = (s-1)^{-1} \sum_{n=0}^{\infty} a_n^n (s-1)^n,
\]

where \( |C_n| \leq C_2 \), \( |a_n^n| \leq C_4 \), \( C_2 > 1 \), \( C_4 > 1 \).

Hence \( C_n^n \) is less than the coefficient of \( (s-1)^{n-1} \) in

\[
\sum_{n=0}^{\infty} C_n (s-1)^n = (1 - C_2 (s-1))^{-1} = \sum_{n=0}^{\infty} (\frac{k+n-1}{n!}) (C_2 (s-1))^n.
\]

Hence

\[
m^1 a_n^n = \sum_{k=n}^{\infty} C_k (s-1)^k \quad \leq \sum_{k=n}^{\infty} C_k \frac{(k+n-1)!}{(k-1)! n!} = C_n \frac{\Gamma(\delta+2)}{(\delta-1)!}\n
= k C_4 \frac{(\delta-2)!}{(\delta-1)!} < e^{C_n}.
\]

**Lemma b.**

\[
\frac{1}{\pi} \int_{1}^{c+iT} |\Gamma(s)\zeta(s-1)| e^{-\text{Re}s} ds > \sum_{n=1}^{\infty} \frac{1}{n} \int_{1}^{c+iT} \frac{|\Gamma(s)\exp(-\text{Re}s)|}{|\zeta(s-1)|} e^{-\text{Re}s} ds - \exp(C_k \log k).
\]

\[ \text{[Titchmarsh (6).]} \]
Hence \[
\sum_i \sum_{m=1}^{\infty} d(m) \frac{\psi(m)}{m} \frac{e^{-m} \sin \frac{m}{r}}{m} > \frac{C_1}{\delta} \sum_{n=1}^\infty \frac{d(n)}{n^\alpha} e^{-n \sin \frac{n}{r}}.
\]

Also, using (7.14.4),
\[
|\Sigma_1| < C_1 \sum_{n=1}^\infty \frac{d(n)}{n^\alpha} \frac{\psi(n)}{n} e^{-n \sin \frac{n}{r}}< C_1 \sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}}
\]

\[\leq C_1 \sum_{n=1}^\infty \frac{e^{-n} \sin \frac{n}{r}}{n} \sum_{m=1}^{n-1} \frac{d(m)e^{-m} \sin \frac{m}{r}}{m}
\]

\[< C_1 \sum_{n=1}^\infty \frac{e^{-n} \sin \frac{n}{r}}{n} \sum_{m=1}^{n-1} \frac{d(m)e^{-m} \sin \frac{m}{r}}{m} < C_1 \log \frac{1}{\delta} \sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}}.
\]

This proves the lemma.

**Lemma 2** \( \beta > 0 \)
\[
\exp \left[ \frac{k}{(\log k)^{\alpha}} \right] < F(k) < \exp \left( C_4 \log \frac{k}{\log k} \right).
\]

It is clear from (8.9.6) that
\[
f_{1}(z) < (1 - \psi)z^{a} \quad (0 < z < 1).
\]

Also, it is easily verified that
\[
\frac{(k + m - 1)!}{(k - 1)!m!} \leq \frac{(k + m - 1)!}{(k - 1)!m!}
\]

Then, for \( 0 < z < 1 \),
\[
f_{1}(z) < \sum_{k=1}^{\infty} \frac{(k + m - 1)!}{(k - 1)!m!}z^{m} = (1 - z)^{-m}.
\]

Hence
\[
\log F_{2}(z) = \sum_{\nu \in \mathbb{C}_{\nu}} \log \log \frac{1}{(p^{-\nu} - 1)z} + \sum_{\nu \in \mathbb{C}_{\nu}} \log f_{2}(p^{-\nu})
\]

\[< \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - z^{-\nu}) + \sum_{\nu \in \mathbb{C}_{\nu}} \log f_{2}(p^{-\nu})
\]

\[< \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - z^{-\nu}) + \sum_{\nu \in \mathbb{C}_{\nu}} \log f_{2}(p^{-\nu})
\]

\[= O(k^{\alpha}(1 - z^{-1})^{2}) + O(k\log k^{1+\epsilon}) = O(k\log k^{1+\epsilon}).
\]

On the other hand, (8.10.2) gives
\[
\log F_{2}(z) > 2\beta \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - \nu^{-1})^{2} - \sum_{\nu \in \mathbb{C}_{\nu}} \log z
\]

\[> 2\beta \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - \nu^{-1})^{2} - \sum_{\nu \in \mathbb{C}_{\nu}} \log z
\]

\[> \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - \nu^{-1})^{2} - \sum_{\nu \in \mathbb{C}_{\nu}} \log z
\]

\[= \sum_{\nu \in \mathbb{C}_{\nu}} \log (1 - \nu^{-1})^{2} - \sum_{\nu \in \mathbb{C}_{\nu}} \log z
\]

\[> C_{4} \log \frac{k}{\log k}^{1+\epsilon}.
\]

Taking
\[z = \left( \frac{C_{4}}{C_{1}} \right)^{1+\epsilon}
\]

the other result follows.

**Proof of Theorem 8.12 for \( \frac{1}{2} < \alpha < 1 \).** It follows from Lemmas 3 and 5 and Stirling's theorem that
\[
\int \frac{\exp \left( \frac{k}{(\log k)^{\alpha}} \right) \exp \left( -k^{\alpha} \right)}{z^{\alpha} - 1} \exp \left( -k^{\alpha} \right) \exp \left( C_{4} \log \frac{k}{\log k} \right)
\]

\[< C_{4} \log \frac{1}{\delta} \sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}}.
\]

Now, if \( 0 < z < 2\pi - 1 \),
\[
\sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}} = F(2\pi - 1) - \sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}}
\]

\[> F(2\pi - 1) - C_{1} \sum_{n=1}^\infty \frac{d(n)e^{-n} \sin \frac{n}{r}}{n^{\beta}}
\]

\[= F(2\pi - 1) - C_{1} \exp \left( \frac{k}{(\log k)^{\alpha}} \right) - C_{4} \log \frac{k}{\log k}^{1+\epsilon}.
\]

\[= F(2\pi - 1) - C_{1} \exp \left( \frac{k}{(\log k)^{\alpha}} \right) - C_{4} \log \frac{k}{\log k}^{1+\epsilon}.
\]

Let \( \delta = \exp \left( C_{4} \log \frac{k}{\log k} \right) \).

Then
\[
\int \frac{\exp \left( \frac{k}{(\log k)^{\alpha}} \right) \exp \left( -k^{\alpha} \right)}{z^{\alpha} - 1} \exp \left( -k^{\alpha} \right) \exp \left( C_{4} \log \frac{k}{\log k} \right)
\]

\[< C_{4} \log \frac{1}{\delta} \exp \left( C_{4} \log \frac{k}{\log k} \right). \]
Suppose now that

\[ |(\alpha + i t)| \leq \exp(\log^2 t) \quad (t > \tau)
\]

where \(0 < \alpha < 1\). Then

\[
\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{it\alpha x} e^{2\pi i t x} \frac{dt}{1 + t^2} \right|^2 dx \leq C_\alpha^2 \int e^{2\pi \log \alpha - \log t} dt.
\]

If \(t > \varepsilon / \delta\), \(k > 4_\alpha\), then

\[
\frac{k}{\delta} < \varepsilon < \frac{t}{2 \log^2 t}
\]

Hence

\[
\int_{\mathbb{R}} e^{2\pi \log \alpha - \log t} \frac{dt}{1 + t^2} \leq \varepsilon \int e^{2\pi \log \alpha - \log t} dt + \int e^{-2\pi \log \alpha - \log t} dt
\]

Hence

\[
\left( \frac{k}{\log k} \right)^\alpha = O\left( \frac{k}{\log k} \right) = O(1),
\]

Hence

\[
\frac{k}{\alpha} \approx 1 + \frac{2\varepsilon}{\delta}
\]

and since \(\varepsilon\) may be as small as we please

\[
\frac{k}{\alpha} \approx 1 + \frac{\varepsilon}{\delta} > 1 - \alpha,
\]

The case \(\alpha = 1\). Suppose that

\[ |(1 + it)| = O(\exp(\log^2 t)) \quad (0 < \beta < \frac{1}{2}).
\]

Then the function

\[ f(t) = \langle e^{-it\alpha} \rangle (0 < \beta < \frac{1}{2})
\]

is bounded on the line \(\alpha = 1\), \(\beta = 1\), \(t > \tau\), and it is \(O(1)\) uniformly in this strip. Hence by the Phragmén–Lindelöf theorem, \(f(t)\) is bounded in the strip, i.e.

\[ |(1 + it)| = O(\exp(\log^2 t))
\]

for \(\frac{1}{2} < \alpha < 2\). Since this is not true for \(\frac{1}{2} < \alpha < 1 - \beta\), it follows that \(\beta > \frac{1}{2}\).

NOTES FOR CHAPTER 8

8.18. Levinson [1] has sharpened Theorems 8.9(A) and 8.9(B) to show that the inequalities

\[ \langle 1 + it \rangle \geq e^\gamma \log t + O(1) \]

and

\[ \frac{1}{\langle 1 + it \rangle} \leq \frac{e^\gamma}{z} (\log \log t - \log \log \log t + O(1)) \]

each hold for arbitrarily large \(t\). Theorem 8.12 has also been improved, by Montgomery [3]. He showed that for any \(\alpha\) in the range \(\frac{1}{2} < \alpha < 1\), and for any real \(\beta\), there are arbitrarily large \(t\) such that

\[ \Re \langle e^{it\log (\alpha + it)} \rangle > \frac{1}{2} (\alpha - \frac{1}{2}) (\log t)^{-\beta} \]

Here \(\langle \alpha \rangle\) is, as usual, defined by continuous variation along lines parallel to the real axis, using the Dirichlet series (1.1.9) for \(\alpha > 1\). It follows in particular that

\[ \langle \alpha + it \rangle = \Omega \left( \left( \frac{1}{2} + \frac{1}{2} \log \log t \right)^{-\beta} \right) \quad (\frac{1}{2} < \alpha < 1),
\]

and the same for \(\langle \alpha + it \rangle^{-1}\). For \(\alpha - \frac{1}{2}\) the best result is due to Balasubramanian and Ramachandra [2], who showed that

\[ \max_{T < x < 2T} \langle \{ \alpha + it \} \rangle > \exp \left( \frac{1}{2} \left( \log H \right)^{-\beta} \right) \]

if \(\log T < H < T\) and \(T > T(\delta)\), where \(\delta\) is any positive constant. Their method is akin to that of §8.12, in that it depends on a lower bound for a mean value of \(\langle \xi(\alpha + it) \rangle\), uniform in \(\alpha\). By contrast, the method of Montgomery [3] uses the formula

\[ \sum_{x} \frac{\Lambda(n)}{n^s} \frac{x}{n^\beta} \int e^{-it\log x} \left( \left( \frac{\sin \frac{1}{2} \log y}{\frac{1}{2} \log y} \right)^{1 + \cos(\beta + y \log x)} \right) dy
\]

\[ = \sum_{n \leq x} \Lambda(n) n^{-s} \left( \left( \frac{1}{2} - \log \frac{1}{2} \right)^{-\beta} + O(x \log t)^{-\beta} \right). \quad (8.12.1)
\]

This is valid for any real \(x\) and \(\beta\), providing that \(\langle \alpha \rangle > 0\) for \(\Re(s) > \alpha\) and \(|s - 1| < 2(\log t)^{\beta}\). After choosing \(x\) suitably one may use the extended version of Dirichlet's theorem given in §8.2 to show that the real part of the sum on the right of (8.12.1) is large at points \(n \leq x \leq T\), spaced at least \(4 \log T\) apart. One can arrange that \(N\) exceeds \(N(e, T)\), whence at least one such \(i_n\) will satisfy the condition that \(\langle \alpha \rangle = 0\) in the corresponding rectangle.
IX

THE GENERAL DISTRIBUTION OF THE ZEROS

9.1. In § 2.12 we deduced from the general theory of integral functions that \( \xi (t) \) has an infinity of complex zeros. This may be proved directly as follows.

We have

\[
\frac{1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots}{\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots} = \frac{1}{4} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{3}{4}.
\]

Hence for \( \sigma > 2 \)

\[
|\zeta (s)| \leq 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \leq 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots < \frac{3}{4},
\]

and

\[
|\zeta (s)| \geq 1 - \frac{1}{2^3} - \frac{1}{3^3} - \frac{1}{4^3} - \cdots > \frac{1}{4}.
\]

Also

\[
R(\zeta (s)) = 1 + \frac{\cos (\log 2)}{2^3} + \cdots > 1 - \frac{1}{2^3} - \cdots > \frac{1}{4}.
\]

Hence for \( \sigma > 2 \) we may write

\[
\log \zeta (s) = \log |\zeta (s)| + i \arg \zeta (s),
\]

where \( \arg \zeta (s) \) is the value of \( \arg \zeta (s) \) in the analytic continuation of the above function along the straight line \( (\sigma - i, \sigma + i) \), provided that \( \zeta (s) \neq 0 \) on this segment of line.

It is clear that

\[
\log |\zeta (s)| < A \quad (\sigma > 2).
\]

For \( \sigma < 2, \) if \( t \neq 0 \), we define \( \log |\zeta (s)| \) as the analytic continuation of the above function along the straight line \( (0, 0, \sigma - i) \). Suppose that \( |t| = 0 \) in or on \( C_4 \). Then \( \log |\zeta| \), defined as above, is regular in \( C_4 \). Let \( M_t \), \( M_t \), \( M_t \) be the maximum modulus on \( C_4 \), \( C_4 \), and \( C_4 \) respectively.

Since \( \zeta (s) = O (t^2), \) \( R(\log (t)) = A \log t \) in \( C_4 \), and the Borel-Carathéodory theorem gives

\[
M_t < \frac{2M_1}{\delta - \delta} A \log T + \frac{M_4}{\delta - \delta} \log (3 + 3T) < A \log T.
\]

Also \( M_t < A \), by (9.1.4). Hence Hadamard's three-circles theorem, applied to the circles \( C_4, C_4, C_4 \), gives

\[
M_t < M_t M_t < A \log P T.
\]

where

\[
1 - \alpha = \beta = \log 4 / \log 3 < 1.
\]

Hence

\[
\zeta (-1 + i T) = O (\exp (\log P T)) = O (P^T).
\]

But by (9.1.2), and the functional equation (2.1.1) with \( \nu = 2 \),

\[
|\zeta (-1 + i T)| > A T^\delta.
\]

We have thus obtained a contradiction. Hence every such circle \( C_t \) contains at least one zero of \( \zeta (s) \), and so there are an infinity of zeros. The argument also shows that the gaps between the ordinates of successive zeros are bounded.

9.2. The function \( N(T) \). Let \( T > 0 \), and let \( N(T) \) denote the number of zeros of the function \( \zeta (s) \) in the region \( 0 < \sigma < 1, 0 < t < T \).

The distribution of the ordinates of the zeros can then be studied by means of formulae involving \( N(T) \).

The most easily proved result is

Theorem 9.2. As \( T \to \infty \)

\[
N(T + 1) - N(T) = O (\log T).
\]

(9.2.1)

For it is easily seen that

\[
N(T + 1) - N(T) \ll n(t^2),
\]

where \( n(t) \) is the number of zeros of \( \zeta (s) \) in the circle with centre \( 3 + i T \) and radius \( T \). Now, by Jensen's theorem,

\[
\int \frac{n(t)}{t} \, dt < \frac{1}{2} \log |(3 + 3T + 3i 0)| \, dt - \log |(3 + 3T)|.
\]

Since \( |(3 + 3T)| < t^4 \) for \( -1 < \sigma < 5 \), we have

\[
\log |(3 + 3T + 3i 0)| < A \log T.
\]

Hence

\[
\int \frac{n(t)}{t} \, dt < A \log T + A < A \log T.
\]

Since

\[
\int \frac{n(t)}{t} \, dt \geq \frac{1}{2} \int \frac{n(t)}{t} \, dt \geq n(\delta) \frac{1}{2} \int \frac{1}{t} \, dt = n(\delta),
\]

the result (9.2.1) follows.

Naturally it also follows that

\[
N(T + 2) - N(T) = O (\log T)
\]

for any fixed value of \( K \). In particular, the multiplicity of a multiple zero of \( \xi (s) \) in the region considered is at most \( O (\log T) \).
9.3. The closer study of $N(T)$ depends on the following theorem.

If $T$ is not the ordinate of a zero, let $N(T)$ denote the value of

$$-\frac{1}{2\pi} \arg \zeta(1+it).$$

obtained by continuous variation along the straight lines joining $2, 2+iT, 1/2+iT$, starting with the value 0. If $T$ is the ordinate of a zero, let $N(T) = S(T) + O(1)$.

Let

$$L(T) = \frac{1}{2\pi} \log \frac{T}{2\pi} - \frac{1}{2} \log \log T - \frac{1}{2} \log \log \log T - \frac{1}{2}.$$ (9.3.1)

**Theorem 9.3.** As $T \to \infty$

$$N(T) = L(T) + S(T) + O(1/T).$$ (9.3.2)

The number of zeros of the function $\zeta(s)$ (see § 2.1) in the rectangle with vertices at $s = \pm T \pm \frac{1}{2} + it$ is $N(T)$, so that

$$2N(T) = \frac{1}{2\pi i} \int_{T}^{T+iT} \frac{\zeta'(s)}{\zeta(s)} ds$$

taken round the rectangle. Since $\zeta(s)$ is even and real for real $s$, this is equal to

$$\frac{1}{2\pi i} \int_{T}^{T+iT} \frac{\zeta'(s)}{\zeta(s)} ds = \frac{1}{\pi i} \int_{T}^{T+iT} \frac{\zeta'(s)}{\zeta(s)} ds,$$

where $\Delta$ denotes the variation from $T$ to $2T+iT$, and hence to $1/2+iT$, along straight lines. Rescaling that

$$\tau(s) = \frac{1}{2}(s-1)(1-s)\log(1-\sigma),$$

we obtain

$$N(T) = \Delta \arg \zeta(1+it) - \Delta \arg (s-1)(1-s)\log(1-\sigma).$$

Now

$$\Delta \arg \zeta(1+it) = \arg(-1) = -\pi,$$

$$\Delta \arg (s-1)(1-s)\log(1-\sigma) = -\frac{1}{2}\pi \log \sigma,$$

and by (4.11.1)

$$\Delta \arg \Gamma(s) = -\frac{3}{2}\pi \log \sigma.$$ (9.4.1)

Adding these results, we obtain the theorem, provided that $T$ is not the ordinate of a zero. If $T$ is the ordinate of a zero, the result follows from

† Backlund (D), (3).
We deduce

**Theorem 9.5.** As \( T \to \infty \)

\[
S(T) = O(\log T),
\]

i.e.

\[
N(T) = \frac{1}{2\pi i} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T).
\]

We apply the lemma with \( f(x) = |S(x)|^2 \), \( \alpha = 0 \), \( \beta = \frac{1}{2} \), and (9.4.2) follows, since \( \xi(0) = O(\zeta(T)) \). Then (9.4.2) follows from (9.3.2). Theorem 9.4 has a number of interesting consequences. It gives another proof of Theorem 9.2, since \( 0 < \theta < 1 \)

\[
L(T + 1) - L(T) = L'(T + \theta) - O(\log T).
\]

We can also prove the following result.

If the zeros \( \beta + iy_k \) of \( L(s) \) with \( y_k > 0 \) are arranged in a sequence \( \rho_\alpha = \beta_\alpha + iy_\alpha \) so that \( y_\alpha > y_\alpha' \), then as \( n \to \infty \)

\[
|\rho_\alpha| \sim y_\alpha \sim \frac{2\pi n}{\log n}.
\]

We have

\[
N(T) \sim \frac{1}{2\pi i} T \log T.
\]

Hence

\[
2\pi n(y_\alpha + 1) \sim (y_\alpha + 1) \log (y_\alpha + 1) \sim y_\alpha \log y_\alpha.
\]

Also

\[
N(y_\alpha - 1) \leq n \leq N(y_\alpha + 1).
\]

Hence

\[
n \sim y_\alpha \log y_\alpha.
\]

and so

\[
y_\alpha \sim \frac{2\pi n}{\log n}.
\]

Also \( |\rho_\alpha| \sim y_\alpha \), since \( \beta_\alpha = 0 \).

We can also deduce the result of \( \| \cdot \| \) that the gaps between the ordinates of successive zeros are bounded. For \( |S(0)| \leq C \log t (t \geq 2) \),

\[
N(T + H) - N(T) = \frac{1}{2\pi i} \int_H^T \log \frac{1}{2\pi} dt + S(T + H) - S(T) = O\left(\frac{1}{H}\right)
\]

\[
\geq \frac{H}{2\pi} \log T - C(\log T + \log H) + O\left(\frac{1}{H}\right),
\]

which is ultimately positive if \( H \) is a constant greater than \( 4\pi C \).

The behavior of the function \( N(T) \) appears to be very complicated. It must have a discontinuity \( k \) where \( T \) passes through the ordinate of a zero of \( L(s) \) of order \( k \) (since the term \( O(1/T) \) in the above theorem is in fact continuous). Between the zeros, \( N(T) \) is constant, so that the variation of \( S(T) \) must just neutralize that of the other terms. In the formula (9.2.1), the term \( \frac{1}{2\pi} \) is presumably overwhelmed by the variations of \( S(T) \). On the other hand, in the integrated formula

\[
\int_0^T S(t) dt = \int_0^T L(t) dt + \int_0^T S(t) dt + O(\log T)
\]

the term in \( S(T) \) certainly plays a much smaller part, since, as we shall presently prove, the integral of \( N(t) \) over \( (0, T) \) is still only \( O(\log T) \). Presumably this is due to frequent variations in the sign of \( S(t) \). Actually we shall show that \( S(t) \) changes sign an infinity of times.

### 9.5. A problem of analytic continuation

The above theorems on the zeros of \( L(s) \) lead to the solution of a curious subsidiary problem of analytic continuation.† Consider the function

\[
P(s) = \sum \frac{1}{n^s},
\]

This is an analytic function of \( s \), regular for \( s > 1 \). Now by (1.8.1)

\[
\mathcal{P}(s) = \sum_{\nu=1}^{\infty} \frac{n(\nu)}{\nu^s} \; \log \; \xi(\nu).
\]

As \( \nu \to \infty \), \( \log \xi(\nu) \sim 2^\nu \log \nu \). Hence the right-hand side represents an analytic function of \( s \), regular for \( s > 0 \), except at the singularities of individual terms. These are branch points arising from the poles and zeros of the functions \( \xi(\nu) \); there are an infinity of such points, but they have no limit-point in the region \( s > 0 \). Hence \( \mathcal{P}(s) \) is regular for \( s > 0 \), except at certain branch points.

Similarly the function

\[
Q(s) = -\frac{\mathcal{P}(s)}{s} = \sum_{\nu=1}^{\infty} \frac{n(\nu)}{\nu^s} \frac{\xi'(\nu)}{\xi(\nu)}
\]

is regular for \( s > 0 \), except at certain simple poles.

We shall now prove that the line \( s = 0 \) is a natural boundary of the functions \( P(s) \) and \( Q(s) \).

We shall in fact prove that every point of \( s = 0 \) is a limit-point of poles of \( Q(s) \). By symmetry, it is sufficient to consider the upper half-plane. Thus it is sufficient to prove that for every \( u > 0 \), \( \delta > 0 \), the square

\[
0 < \sigma < \delta, \quad u < t < u + \delta
\]

c containing at least one pole of \( Q(s) \).

† Landau and Weil (1).
General distribution of zeros

As \( p \to \infty \) through primes,

\[
N(p(u \pm b)) \sim \frac{1}{2\pi} (v \pm b) \log p, \quad N(pu) \sim \frac{1}{2\pi} w \log p,
\]

by Theorem 9.4. Hence for all \( p \geq p_0(\delta, \alpha) \)

\[
N(p(u \pm b)) - N(pu) > 0.
\] (9.2.5)

Also, by Theorem 9.2, the multiplicity \( m(p) \) of each zero \( \gamma = \beta + iy \) with \( \delta \leq y \leq 2 \) is less than \( A \log y \), where \( A \) is an absolute constant.

Now choose \( p = p(\delta, \alpha) \) satisfying the conditions

\[
p > \frac{1}{\gamma}, \quad p > \frac{2}{\delta}, \quad p \geq p_0(\delta, \alpha), \quad p > A \log(p(u \pm b)).
\]

There is then, by (9.2.5), a zero \( \gamma \) of \( \zeta(s) \) in the rectangle

\[
\frac{1}{4} \leq \delta < 1, \quad pu < \gamma < p(u \pm b).
\] (9.5.6)

Since \( \gamma > pu > 2 \), its multiplicity \( m(\gamma) \) satisfies

\[
v(\gamma) < A \log y < A \log(p(u \pm b)) < p,
\]

and so is not divisible by \( p \).

The point \( \gamma/p \) belongs to the square (9.5.4). We shall show that this point is a pole of \( \zeta(s) \). Let \( m \) run through the positive integers (finite in number) for which \( \gamma/m(\gamma/p) = 0 \). Then we have to prove that

\[
\sum_{m = 1}^{\infty} \frac{\mu(m)}{m} \left( \frac{m(\gamma/p)}{p} \right) > 0.
\] (9.5.7)

The term of this sum corresponding to \( m = p \) is \(-\epsilon(p)/p \). No other \( m \)

occurring in the sum is divisible by \( p \), since for \( m \geq 2p \)

\[
\frac{\mu(m)}{m} \left( \frac{m(\gamma/p)}{p} \right) = \frac{\mu(m)}{m} \left( \frac{m(\gamma/p)}{p} \right).
\]

Hence

\[
\sum_{m = 1}^{\infty} \frac{\mu(m)}{m} \left( \frac{m(\gamma/p)}{p} \right) = \frac{\mu(\gamma/p)}{p}.
\]

where \( a \) and \( b \) are integers, and \( p \) is not a factor of \( b \). Since \( p \) is also not a factor of \( \epsilon(p) \), \( \gamma/p \) is a pole of \( \zeta(s) \).

There are various other functions with similar properties. For example, let

\[
f_{k,l}(s) = \sum_{n \geq 1} \frac{\delta_{k,l}(n)}{n^s},
\]

where \( k \) and \( l \) are positive integers, \( k \geq 2 \). By (1.2.2) and (1.2.10),

\( f_{k,l}(s) \) is a meromorphic function of \( s \) if \( l = 1 \), or if \( l = 2 \) and \( k = 2 \).

For all other values of \( k \) and \( l \), \( f_{k,l}(s) \) has \( s = 0 \) as a natural boundary, and it has no singularities other than poles in the half-plane \( s > 0 \).

\( \uparrow \) Remarks (1).
it follows that
\[ \sum_{p < t \leq 1} \frac{1}{t - \sigma} - \frac{1}{t} = O(\log t). \]
Similarly
\[ \sum_{t < p < t + \frac{1}{2}} \frac{1}{t} - \frac{1}{t + \frac{1}{2}} = O(\log t) \]
and the result follows again.

The corresponding formula for \( \log \zeta(s) \) is given by

**Theorem 9.6 (B).** We have

\[
\log \zeta(t) = \sum_{t < \gamma < s} \log(t - \sigma) + O(\log t).
\]  
(9.6.3)

uniformly for \(-1 \leq \sigma \leq 2\), where \( \log \zeta(s) \) has its usual meaning, and \(-\sigma < 1 \log(t - \sigma) \leq \sigma\).

Integrating (9.6.1) from \( s \) to \( 2 + i t \), and supposing that \( t \) is not equal to the ordinate of any zero, we obtain

\[
\log \zeta(t) - \log \zeta(2 + it) = \sum_{s < \gamma < (2 + it)} (\log(t - \sigma) - \log(2 + it - \sigma)) + O(\log t).
\]

Now \( \log \zeta(2 + it) \) is bounded; also \( \log(2 + it - \sigma) \) is bounded, and there are \( O(\log t) \) terms. Their sum is therefore \( O(\log t) \). The result therefore follows for such values of \( t \), and then by continuity for all values of \( s \) in the strip other than the zeros.

9.7. As an application of Theorem 9.6 (B) we shall prove the following theorem on the minimum value of \( \zeta(s) \) in certain parts of the critical strip. We know from Theorem 8.12 that \( \zeta(s) \) is sometimes large in the critical strip, but we can prove little about the distribution of the values of \( s \) for which it is large. The following result states a somewhat weaker inequality, but states it for many more values of \( t \).

**Theorem 9.7.** There is a constant \( A \) such that each interval \( (T, T + 1) \) contains a value of \( t \) for which

\[
|\log \zeta(t)| > t^{-A} \quad ( -1 \leq \sigma \leq 2 ).
\]  
(9.7.1)

Further, if \( H \) is any number greater than unity, then

\[
|\log \zeta(t)| > T^{-H} \quad (9.7.2)
\]

for \(-1 \leq \sigma \leq 2\), \( T \leq t \leq T + 1 \), except possibly for a set of values of \( t \) of measure \( 1/H \).

Taking real parts in (9.6.3),

\[
\log \zeta(t) = \sum_{t < \gamma < 1} \log(t - \sigma) + O(\log t) \geq \sum_{t < \gamma < 1} \log(t - \gamma) + O(\log t).
\]  
(9.7.3)

\[ \dagger \] Valiron (1), Landau (8), (18), Hohenzelt (3).
The first term on the left is equal to
\[
\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^s \Gamma(s) \, ds = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-\pi x^2} \Gamma(s) \, ds
\]
\[
= -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\pi x^2} x^s \Gamma(s).
\]
Evaluating the other integral in the same way, and multiplying through by \(v\), we obtain Ramanujan's result
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\pi x^2} x^s \Gamma(s) = \frac{1}{2\pi i} \sum_{\nu} \Gamma(\frac{1}{2} - \nu) \phi(\nu).
\]  
(9.8.1)

We have, of course, not proved that the series on the right is convergent in the ordinary sense. We have merely proved that it is convergent if the terms are bracketed in such a way that two terms for which
\[
|\nu - \nu'| < \exp(-A_1 y)(y) + \exp(-A_1 y')/\log y'
\]
are included in the same bracket. Of course the zeros are, on the average, much farther apart than this, and it is quite possible that the series may converge without any bracketing. But we are unable to prove this, even on the Riemann hypothesis.

9.9. We next prove a general formula concerning the zeros of an analytic function in a rectangle.† Suppose that \(\phi(s)\) is meromorphic in and upon the boundary of a rectangle bounded by the lines \(t = 0, t = T, \sigma = a, \sigma = \beta \geq a\), and regular and not zero on \(\sigma = \beta\). The function \(\log \phi(s)\) is regular in the neighborhood of \(s = \beta\) and hence starting with any one value of the logarithm, we define \(F(s) = \log \phi(s)\).

For other points \(s\) of the rectangle, we define \(F(s)\) to be the value obtained from \(\log \phi(s)\) by continuous variation along \(s = \sigma + it\) constant from \(\beta + it\) to \(\sigma + it\), provided that the path does not cross a zero or pole of \(\phi(s)\); if it does, we put
\[F(s) = \lim_{\epsilon \to 0} F(s + \epsilon + it) \).

Let \(\sigma(T)\) denote the excess of the number of zeros over the number of poles in the part of the rectangle for which \(\sigma > \sigma\), including zeros or poles on \(t = T\), but not those on \(t = 0\),
\[
\sigma(T) = \int_{0}^{\infty} F(s) \, ds = -2\pi \int_{-T}^{T} \log \phi(T) \, ds.
\]
(9.9.1)

We may suppose \(t = 0\) and \(t = T\) to be free from zeros and poles of \(\phi(s)\); it is easily verified that our conventions then assure the truth of the theorem in the general case. We have
\[
\int_{0}^{\infty} F(s) \, ds = \int_{0}^{\infty} F(\beta + it) \, ds = \int_{0}^{\infty} \{F(\beta + it) - F(\beta + it)\} \, dt.
\]
(9.9.2)

The last term is equal to
\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{\psi(s + \eta i)}{\phi(s + \eta i)} \, ds \, d\eta
\]
and by the theorem of residues
\[
\int_{0}^{\infty} \frac{\psi(s)}{\phi(s)} \, ds = \left(\int_{0}^{\infty} + \int_{-\infty}^{0} \right) \frac{\psi(s)}{\phi(s)} \, ds - 2\pi i \sum C(e, T)
\]
\[
- \int_{0}^{\infty} \{F(\beta + it) - F(\beta + it)\} \, dt.
\]
(9.9.3)

Substituting this in (9.9.2), we obtain (9.9.1).

We deduce

**Theorem 9.9.** If \(S_1(T) = \int_{0}^{T} S(t) \, dt\),

then
\[
S_1(T) = \frac{1}{2} \int_{0}^{T} \log \|\phi(s) - 1\| \, ds + O(1).
\]
(9.9.3)

Take \(d(a) = \psi(a)\), as in the above formula, and take the next term.

We obtain
\[
\int_{0}^{\infty} \log \phi(s) \, ds = \int_{0}^{\infty} \log \phi(s+it) \, ds + \arg \phi(s) \, dt + \int_{0}^{\infty} \log \phi(s+it) \, ds +\]
\[
+ \arg \phi(\beta + it) \, dt = 0.
\]
(9.9.4)

The term in \(\sigma, T\), being purely imaginary, disappearing. Now make \(\beta \to \infty\). We have
\[
\log \phi(s) = \log(1 + \frac{1}{2} + \cdots) = O(2^{-s})
\]
as \(s \to \infty\), uniformly with respect to \(t\). Hence \(\arg \phi(s) = O(2^{-s})\), so that the second integral tends to 0 as \(\beta \to \infty\). Also the first integral is a constant, and
\[
\int_{0}^{\infty} \log \phi(s+it) \, ds = \int_{0}^{\infty} O(2^{-s}) \, ds = O(1).
\]
Hence the result.

Theorem 9.9 (A). \( S(t) = O(\log T) \).

By Theorem 9.9 (B)
\[
\frac{1}{\pi} \log |\Gamma(s)| \, ds = -\sum_{\rho \in T} \frac{1}{\pi} \log |s - \rho| \, ds + O(\log T).
\]
The terms of the last sum are bounded, since
\[
\log(1 + t) \geq \frac{1}{2} \log((s-\beta)^2 + (y-t)^2) \, ds \geq 2 \frac{1}{2} \log(s-\beta) \, ds > -A.
\]
Hence
\[
\int \frac{1}{\pi} \log |\Gamma(s)| \, ds = O(\log T),
\]
and the result follows from the previous theorem.

It was proved by F. and R. Nevanlinna (1) that
\[
\int \frac{1}{\pi} \log |\Gamma(s)| \, ds = A + O\left(\frac{\log T}{T}\right).
\]
This follows from the previous result by integration by parts, for
\[
\int \frac{1}{\pi} \log |\Gamma(s)| \, ds = \int \frac{S(t)}{\pi} dt = A + S(t) \frac{\log T}{T}.
\]
Since \( S(t) = O(\log T) \), the middle integral is \( O(T \cdot \log T) \), and the last term is
\[
O\left(\frac{1}{\pi} \log T \int \frac{1}{\pi} \log |\Gamma(s)| \, ds\right) - O\left(\frac{\log T}{T} \int \frac{1}{\pi} \log |\Gamma(s)| \, ds\right).
\]
Hence the result follows. A similar result clearly holds for
\[
\int \frac{1}{\pi} \log |\Gamma(s)| \, ds \quad (0 < a < 1).
\]
It has recently been proved by A. Selberg (5) that
\[
S(t) = \Omega_{\text{i}}(\log t (\log \log t)^2)
\]
with a similar result for \( S(t) \); and that
\[
S(t) - \Omega_{\text{i}}(\log t (\log \log t)^2).
\]

9.10. Theorem 9.10. \( S(t) \) has an infinity of changes of sign.
Consider the interval \( (\gamma_i, \gamma_{i+1}) \) in which \( N(t) = n \). Let \( T(t) \) be the 

\[\text{* Littlewood (3).}\]
This would follow at once from (9.3.2) if it were possible to prove that
\( S(t) = o(\log t) \).

The argument given in \( \S \).41 shows that the gaps are bounded. Here we have to apply a similar argument to the strip \( T - \delta < t < T + \delta \), where \( \delta \) is arbitrarily small, and it is clear that we cannot use four concentric circles. But the ideas of the theorems of Borel–Carathéodory and Hadamard are in no way essentially bounded up with sets of concentric circles, and the difficulty can be surmounted by using suitable elongated curves instead.

Let \( D_k \) be the rectangle with centres \( -1+iT \) and a corner at \(-1+i(T+\delta)\), the sides being parallel to the axes. We represent \( D_k \) conformally on the unit circle \( U_k \) in the \( z \)-plane, so that its centre \( 0+iT \) corresponds to \( z = 0 \). By this representation a set of concentric circles \( |z| = \rho \) inside \( U_k \) will correspond to a set of convex curves inside \( D_k \), such that as \( \rho \to 0 \) the curve up on the point \( 0+iT \), while as \( \rho \to 1 \) it tends to coincide with \( D_k \). Let \( D_{k-1}, D_{k+1} \) be circles (independent, of course, of \( T \)) for which the corresponding curves \( U_{k-1}, U_{k+1} \) in the \( z \)-plane pass through the points \( 0+i(T-\delta), 0+i(T+\delta) \) respectively.

The proof now proceeds as before. We consider the function
\[
\tau(t) = \log \left( \frac{\log |t|}{\log T} \right),
\]
where \( s = \tau(t) \) is the analytic function corresponding to the conformal representation; and we apply the theorems of Borel–Carathéodory and Hadamard in the same way as before.

9.12. We shall now obtain a more precise result of the same kind.†

**Theorem 9.12.** For every large positive \( T \), \( \tau(t) \) has a zero \( \beta+iy \) satisfying
\[
|\gamma - T| < \frac{A}{\log \log T}.
\]

This was first proved by Littlewood by a detailed study of the conformal representation used in the previous proof. This involves rather complicated calculations with elliptic functions. We shall give here two proofs which avoid these calculations.

In the first, we replace the rectangles by a succession of circles. Let \( T \) be a large positive number, and suppose that \( \tau(t) \) has no zero \( \beta+iy \) such that \( T - \delta < \gamma < T + \delta \), where \( \delta < 1 \). Then the function
\[
\tau(t) = - \log \left( \frac{\log |t|}{\log T} \right),
\]
where the logarithm has its principal value for \( \sigma > 2 \), is regular in the rectangle
\[-2 < \sigma < 3, \quad T - \delta < t < T + \delta.\]

† Littlewood (3); proved given here by Titchmarsh (12); Krasnovskikh (1).
so that
\[ z = \frac{12}{n-1} \cdot \frac{A}{\log \log z^T} \]
and the result follows.

9.13. Second Proof. Consider the angular region in the \( s \)-plane with vertex at \( s = -3 + iT \), bounded by straight lines making angles \( \pm \frac{\pi}{3} \) with the real axis.

Let
\[ w = (s + 3 - iT)^{\omega \alpha}. \]

Then the angular region is mapped on the half-plane \( R(w) \geq 0 \). The point \( s = 2 + iT \) corresponds to
\[ w = 5^{\omega \alpha}. \]

Let
\[ z = \frac{w - 5^{\omega \alpha}}{w + 5^{\omega \alpha}}. \]

Then the angular region corresponds to the unit circle in the \( z \)-plane, and \( z = 2 + iT \) corresponds to \( z = -r \).

Thus \( \omega \alpha \) has no zeros in the angular region, so that \( \log \omega \alpha \) is regular in it.

Let \( s = \frac{3}{2} + iT', -1 + iT', -2 + iT' \) correspond to \( z = -r_1, -r_2, -r_3 \) respectively. Let \( M_1, M_2, M_3 \) be the maxima of \( \log \omega \alpha(z) \) on the \( s \)-curves corresponding to \( |s| = r_1, r_2, r_3 \). Then Hadamard’s three-circle theorem gives
\[ \log M_k < \log \omega \alpha + \log r_k/r_1 \log M_1, \]

It is easily verified that, on the curve corresponding to \( |s| = r_1, \sigma \geq \frac{3}{2} \). For \( s = \frac{3}{2} + iT' \),
\[ \sigma = -3 + (2 + T')^{\omega \alpha \cos \left[ \frac{\pi}{3} \arctan \frac{3}{T'} \right]}, \]

which is a minimum at \( \sigma = 0 \), for given \( T' \), if \( 0 < \sigma < \frac{3}{2} \); and the minimum is \( -3 + \omega \alpha \cdot \frac{3}{2} \), which, as a function of \( T' \), is a minimum when \( T' \) is a minimum. I.e. when \( z = -r_1 \). It therefore follows that \( \log M_1 < A \).

Since \( R(\log \omega \alpha) < A \log T \) in the angle, it follows from the Borel-Carathéodory theorem that
\[ M_1 < \frac{2}{1 - r_1} \cdot \left( A \log T + A \right) < \frac{A \log T}{1 - r_1}. \]
\[ f(s) = \frac{\log T}{n} \]

and hence

\[ \log \left( \frac{\log T}{n} \right) < \frac{\pi}{\alpha} \log \frac{9}{4} + \log \frac{1}{\alpha} + A < \frac{\pi}{\alpha} \log T. \]

This proves the theorem.

9.15. The function \( N(n, T) \). We define \( N(n, T) \) to be the number of zeros \( \beta + \imath \gamma \) of the zeta-function such that \( \beta > a, 0 < t < T \). For each \( T \), \( N(n, T) \) is a non-increasing function of \( n \), and is \( 0 \) for \( n \gg 1 \). On the Riemann hypothesis, \( N(n, T) \to 0 \) for \( n \gg 1 \). Without any hypothesis, all that we can say so far is that

\[ N(n, T) < N(T) < A T \log T \]

for \( \frac{1}{2} < \sigma < 1 \).

The object of the next few sections is to improve upon this inequality for values of \( n \) between \( \frac{1}{2} \) and 1.

We return to the formula (9.9.1). Let \( \xi(n) = \xi(n), \alpha = \omega, \beta = 2 \), and this time take the imaginary part. We have

\[ x(n, T) = N(n, T) (\sigma < 1), \quad x(n, T) = 0 (\sigma \geq 1). \]

We obtain, if \( T \) is not the ordinate of a zero,

\[ 2 \pi = \int\limits_{\sigma=1}^{\sigma=1} \frac{\log |(\xi(\sigma+iT)\log T) + \lambda \sum_{\gamma} \arg (\sigma+iT) \log T |}{\sigma} d\sigma. \]

where \( \lambda \sum_{\gamma} \) is independent of \( T \). We deduce

**Theorem 9.13.** If \( \frac{1}{2} < \sigma < 1 \), and \( T \to \infty \),

\[ 2 \pi \int\limits_{\sigma=1}^{\sigma=1} \frac{\log |(\xi(\sigma+iT)\log T) \lambda \sum_{\gamma} \arg (\sigma+iT) \log T |}{\sigma} d\sigma = O(\log T). \]

We have

\[ \int\limits_{\sigma=1}^{\sigma=1} \log |(\xi(\sigma+iT)\log T) \lambda \sum_{\gamma} \arg (\sigma+iT) \log T | d\sigma = O(1). \]

Also, by \( \frac{1}{2} \), \( \arg (\alpha+iT) = O(\log T) \) uniformly for \( \sigma > \frac{1}{2} \) if \( T \) is not the ordinate of a zero. Hence the integral involving \( \arg (\alpha+iT) \) is \( O(\log T) \). The result follows if \( T \) is not the ordinate of a zero, and this restriction can then be removed from considerations of continuity.

\[ \text{Littledale (4).} \]
Theorem 9.16 (A). For any fixed $a$ greater than $1$, 
\[ N(a, T) = O(T). \]

For any non-negative continuous $f(t)$
\[ \frac{1}{b-a} \int_a^b \log f(t) \, dt \leq \log \left( \frac{1}{b-a} \int_a^b f(t) \, dt \right). \]

Thus, for $\frac{1}{2} < \alpha < 1$,
\[ \int \log |f(a+it)| \, dt = \frac{1}{\alpha} \int \log |f(a+it)^\alpha| \, dt \]
\[ \leq \frac{1}{\alpha} T \log \left( \frac{1}{b-a} \int_a^b |f(a+it)|^\alpha \, dt \right) = O(T) \]
by Theorem 7.2. Hence, by Theorem 9.15,
\[ \int N(a, T) \, da = O(T) \]
for a $\alpha > \frac{1}{2}$. Hence, if $a_0 = \frac{1}{2} + \log a_1$,
\[ N(a_0, T) \leq \frac{1}{a_0-a_1} \int \log |f(a+it)| \, dt \leq \frac{1}{a_0-a_1} N(a, T) \, da = O(T), \]
the required result.

From this theorem, and the fact that $N(T) \sim AT \log T$, it follows that all but an infinitesimal proportion of the zeros of $L(s)$ lie in the strip $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$, however small $\delta$ may be.

9.16. We shall next prove a number of theorems in which the O(T) of Theorem 9.15 (A) is replaced by O(T), where $\delta < 1$. We do this by applying the above methods, not to $L(s)$ itself, but to the function
\[ \zeta(s) M(\sigma) = \zeta(s) \sum n^{-s}. \]
The zeros of $\zeta(s)$ are zeros of $\zeta(s) M_2(s)$. If $\sigma > 1$, $M_2(s) \rightarrow 1/(s-1)$ as $X \rightarrow \infty$, so that $\zeta(s) M_2(s) \rightarrow 1$. On the Riemann hypothesis this is also true for $\frac{1}{2} < \sigma < 1$. Of course we cannot prove this without any hypothesis, but we can choose $X$ so that the additional factor neutralizes to a certain extent the oscillations of $\zeta(s)$, even for values of $\sigma$ less than 1.

Let $f_\sigma(t) = \zeta(t) M_2(t) - 1$. 

† Bohr and Landau (4), Littlewood (4).
‡ Bohr and Landau (6), Glaeser (1), Landau (12), Titchmarsh (6), Ingham (8).
so that
\[ \int_1^T \log |h(t+i)| dt \ll \frac{T}{2 \pi} - o(T \log T). \]

Also we can apply the lemma of § 9.6 to \( h(t) \), with \( \alpha = 0, \beta = \frac{1}{4}, m \gg 1, \) and \( M_j = O(X^2 T^{\sigma_4}). \) We obtain
\[ \arg \left( \frac{h(t+it)}{h(t)} \right) \to O(\log T \log \log T), \]

for \( \sigma \gg \frac{1}{4}. \) Hence
\[ \int_1^T \frac{1}{X} \, \arg h(t+iT) \, dt \to O(\log T \log \log T) = O(\log T). \]

Hence
\[ \int_1^T \frac{1}{X} \, \nu(\sigma, \frac{1}{4}, T) \, dt \to O(T \log \log T), \]

Also
\[ \int_1^T \nu(\sigma, \frac{1}{4}, T) \, dt \to \frac{1}{2} N(\sigma, \frac{1}{4}, T) \to (\sigma - \frac{1}{2}) \Re \left( \frac{1}{4} \right) \]

with \( a_0 < \alpha < 2. \) Taking \( \alpha = \sigma_0 + \log T, \) we have
\[ \frac{T}{2 \pi} \log \left( 1 + \frac{1}{T} \right) = O(T \log \log T), \]

Hence
\[ N(\sigma, \frac{1}{4}, T) = O(T \log \log T). \]

Replacing \( T \) by \( \frac{T}{4}, \frac{T}{8}, ... \) and adding, the result follows.

9.17. The simplest application is

**Theorem 9.17.** For any fixed \( a \) in \( \frac{1}{4} < \sigma < 1, \)
\[ N(\sigma, T) = O(T \log \log T). \]

We use Theorem 4.11 with \( x = T, \) and obtain
\[ f_\sigma(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\sigma} - 1 + o(T^{-\sigma} \log T) \]
\[ = \int_1^x \frac{b_\sigma(X)}{X^\sigma} \, dx + o(T^{-\sigma} \log T), \]

where, if \( X < T, \) \( b_\sigma(X) = 0 \) for \( \sigma < X \) and for \( n > XT; \) and, as for \( \sigma < 1, \)
\[ \Re(\frac{1}{4}) = \sigma \leq 1. \]

Hence
\[ \int_1^T \frac{1}{X} \sum_{n} \frac{b_\sigma(X)}{n^\sigma} \, dx = \int_1^T \sum_{n} \frac{\log(n)}{n^\sigma} \, dx + o(T^{-\sigma} \log T) \]
\[ = o(T^{\sigma - \frac{1}{2}} \log T) + O(T \log \log T) \]
\[ = O(T^b \log T \log \log T) \]

by (7.2.1). These terms are of the same order (apart from \( x \)'s) if \( X = T^{1/2}, \) and then
\[ \int_1^T \sum_{n} \frac{b_\sigma(X)}{n^\sigma} \, dx = O(T \log \log T). \]

The \( O \)-term in (9.17.1) gives
\[ O(T^b \log T \log \log T) = O(T^{-\sigma} \log T). \]

The result therefore follows from Theorem 9.16.

9.18. The main instrument used in obtaining still better results for \( N(\sigma, T) \) is the convexity theorem for mean values of analytic functions proved in § 7.8. We require, however, some slight extensions of the theorem. If the right-hand sides of (7.8.1) and (7.8.2) are replaced by finite sums
\[ \sum C(T^{\sigma_1}) + \sum C(T^{\sigma_2}), \]

then the right-hand sides of (7.8.3) is clearly to be replaced by
\[ K \sum C(T^{\sigma_1}) \log T \log \log T \]

In one of the applications a term \( T^b \log T \) occurs in the numerator instead of the above \( T^b \). This produces the same change in the result. The only change up to this point is that, instead of the term
\[ \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx \]

we obtain a term
\[ \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx \]

by (7.2.1). These terms are of the same order (apart from \( x \)'s) if \( X = T^{1/2}, \) and then
\[ \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx = O(T \log \log T). \]

Theorem 9.20. Let \( N(\sigma, T) = O(T^{-\sigma} \log T \log \log T) \]

uniformly for \( \frac{1}{8} \leq \sigma \leq 1. \)

If \( 0 < \delta < 1, \)
\[ \int_1^T f_\delta(x) \, dx = \int_1^T \sum_{n} \frac{\mu(n) \log(n)}{n^{\sigma_1}} \, dx \]
\[ = \int_1^T \sum_{n} \frac{\mu(n) \log(n)}{n^{\sigma_1}} \, dx \]
\[ \leq \int_1^T \sum_{n} \frac{\mu(n) \log(n)}{n^{\sigma_1}} \, dx \]

by (7.2.1). These terms are of the same order (apart from \( x \)'s) if \( X = T^{1/2}, \) and then
\[ \int_1^T \sum_{n} \frac{\mu(n) \log(n)}{n^{\sigma_1}} \, dx = O(T \log \log T). \]

The \( O \)-term in (9.17.1) gives
\[ O(T^b \log T \log \log T) = O(T^{-\sigma} \log T). \]

The result therefore follows from Theorem 9.16.

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uniformly for \( \frac{1}{8} \leq \sigma \leq 1. \)

If \( 0 < \delta < 1, \)
\[ \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx \]
\[ = \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx \]

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9.18. The main instrument used in obtaining still better results for \( N(\sigma, T) \) is the convexity theorem for mean values of analytic functions proved in § 7.8. We require, however, some slight extensions of the theorem. If the right-hand sides of (7.8.1) and (7.8.2) are replaced by finite sums
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by (7.2.1). These terms are of the same order (apart from \( x \)'s) if \( X = T^{1/2}, \) and then
\[ \int_1^T \sum_{n} \frac{1}{n^{\sigma_1}} \log(n) \, dx = O(T \log \log T). \]
Now \[ \sum_{n \leq x} \frac{d(m)(n)}{n^{\sigma}} < A \log x, \]
Hence \[ \sum_{n \leq x} \frac{d(n)}{n^{\sigma}} < A \log x - \sum_{n \leq x} \frac{d(m)(n)}{n^{\sigma}} \log n/m < A \log x. \]

\[ \frac{d(n)}{n^{\sigma}} \int \frac{1 + \frac{x}{n^{\sigma}}}{x^{2 + \frac{x}{n^{\sigma}}}} \, dx = \int \frac{1 + \frac{x}{n^{\sigma}}}{x^{2 + \frac{x}{n^{\sigma}}}} \, dx \]
\[ = \int \frac{x^{1-\sigma}}{x^{2+\sigma}} \, dx = \frac{x^{1-\sigma}}{x^{2+\sigma}} \log x. \]
(putting \( x = X n^{\sigma} \))
\[ \frac{d(n)}{n^{\sigma}} \int \frac{1 + \frac{x}{n^{\sigma}}}{x^{2 + \frac{x}{n^{\sigma}}}} \, dx \]
\[ = \frac{x}{x^{2+\sigma}} \log x \]
Hence \[ \sum_{n \leq x} \frac{d(n)}{n^{\sigma}} < \frac{A \log x}{X^{1+\alpha}} < \frac{A}{X^{\alpha}} \]
since \( x^{\alpha} = \exp(-x^{\alpha-\epsilon}) > \frac{1}{4!} (2 \log x)^4 \).

Also, since \( 1 < \log x \lambda + \alpha - 1 < \log x \lambda - 1 \)
for \( \lambda > 1 \),
\[ \sum_{n \leq x} \frac{d(n)}{n^{\sigma}} \log n/m < \sum_{n \leq x} \frac{d(n)}{n^{\sigma}} \log n/m \]
\[ < \frac{\log x}{X^{1+\alpha}} \sum_{n \leq x} \frac{d(n)}{n^{\sigma}} \log n/m \]
Hence \[ \int \int \frac{d(n)(n)}{n^{\sigma}} \, dx < A \left( \frac{1}{x^{2+\sigma}} \right)^{-1}. \]

(9.18.11)
For \( \sigma = 1 \) we use the inequalities \( \log \Gamma(z) \leq z \log z + \frac{1}{2} \log z + (z - 1) \log 2 \pi + 1 \).

\[ \int \frac{d(n)(n)}{n^{\sigma}} \, dx < A \left( \frac{1}{x^{2+\sigma}} \right)^{-1}. \]

(9.18.12)
Now \( \log \Gamma(z) \leq z \log z + \frac{1}{2} \log z + (z - 1) \log 2 \pi + 1 \).
Hence \[ \int \frac{d(n)(n)}{n^{\sigma}} \, dx < A \left( \frac{1}{x^{2+\sigma}} \right)^{-1}. \]

(9.18.13)
From (9.18.1), (9.19.1), and the convexity theorem, we obtain
\[
\int_0^\infty \left| f_2(\sigma + it) \right|^2 dt < \frac{1}{T} \left( \frac{T(\sigma + 1)}{\sigma + 2} \right)^{1/2} \left( \frac{T(\sigma + 1)}{\sigma + 2} \right)^{1/2} \left( \frac{T(\sigma + 1)}{\sigma + 2} \right)^{1/2} \left( \frac{T(\sigma + 1)}{\sigma + 2} \right)^{1/2}.
\]

If \( X = T \), \( \delta = 1/\log(T+2) \), the result follows as before.

This is an improvement on Theorem 9.17 if \( \frac{1}{2} < \sigma < \frac{3}{4} \).

Various results of this type have been obtained,† the most successful being

Theorem 9.19 (B). \( N(\sigma, T) = O(T^{\frac{1}{2}-\epsilon} \log T) \).

This depends on a two-variable convexity theorem,‡ if
\[
\tilde{J}(\alpha, \lambda) = \int \left| f(\alpha + it) \right|^2 dt
\]
then
\[
\tilde{J}(\alpha, \beta, \gamma) = O(T^{\frac{1}{2}-\epsilon} \log T)
\]
\( \sigma < \alpha < \beta \).

We have
\[
\int_0^\infty \left| f_2(\sigma + it) \right|^2 dt < \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt.
\]

For \( T \to \infty \). If the zeros of \( \zeta(s) \) are \( \beta + i\gamma \), this is equal to
\[
\int \left| f(\sigma + it) \right|^2 dt = \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt + \int \left| f(\sigma + it) \right|^2 dt.
\]

Hence an equivalent problem is that of the sum
\[
\sum_{\gamma > 1} \beta - \frac{1}{2}.
\]

There are some immediate results,† if we apply the above argument, but use Theorem 7.2 (A) instead of Theorem 7.2, we obtain at once
\[
\int \left| f(\sigma + it) \right|^2 dt < A T^{\frac{1}{2}-\epsilon} \log \left( \min \left( \log T, \log \frac{1}{\sigma - \frac{1}{2}} \right) \right).
\]

These, however, are superseded by the following analysis, due to A. Selberg (2), the principal result of which is that
\[
\int N(\sigma, T) dt = O(T).
\]

We consider the integral
\[
\int \left| f(\sigma + it) \right|^2 dt,
\]
† Titchmarsh (3), Ingham (5, 6).
‡ Ingham (6).
§ Gabriel (1).
where \(0 < U < T\) and \(\psi\) is a function to be specified later. We use the formulas of §1.17. Since
\[
e^\theta = (\frac{t}{2\pi})^{1/2} e^{-\frac{t}{2|\log t|}}(1 + O(\frac{1}{e^t})),
\]
we have
\[
Z(t) = z(t) + \varepsilon(t) + O(t^{-1}),
\]
(9.20.6)
where
\[
z(t) = \left(\frac{t}{2\pi}\right)^{1/2} e^{-t} \sum_{n \leq t} n^{-1/2}
\]
and \(\varepsilon(t) = (t/2\pi)^{1/2}\). Let \(T' < t < T + U\), \(r = (T'/2\pi)^{1/2}\), \(r' = (T + U) / 2\pi\).

Let
\[
z(t) = \left(\frac{t}{2\pi}\right)^{1/2} e^{-t/2} \sum_{n \leq t} n^{-1/2}
\]

Proceeding as in § 7.3, we have
\[
\int_T^{T+U} \left[ \varepsilon(t) - z(t) \right]^2 dt = \frac{U}{(\log T)^2} \sum_{r \leq r' \leq T} \frac{1}{(r')^2} + O(T \log T)
\]
(9.20.7)

9.21. Lemma 9.21. Let \(m\) and \(n\) be positive integers, \((m, n) = 1\), \(M = \max(m, n)\). Then
\[
\int_T^{T+U} \frac{z(t) \varepsilon(t)(n^M)}{\min(m, n)^2} dt = \frac{U}{(\log T)^2} \sum_{r \leq r' \leq T} \frac{1}{(r')^2} + O(T M^2 \log(MT)).
\]

The integral is
\[
\sum_{r \leq r' \leq T} \frac{1}{(r')^2} \int_T^{T+U} \frac{n^M}{m^2} dt.
\]

The terms with \(m^2 = \nu^2\) contribute
\[
U \sum_{r \leq r' \leq T} \frac{1}{(r')^2} = U \sum_{r \leq r' \leq T} \frac{1}{(r')^2} = \frac{U}{(\log T)^2} \sum_{r \leq r' \leq T} \frac{1}{(r')^2}.
\]

The remaining terms are
\[
O\left(\sum_{r \leq r' \leq T} \frac{1}{(r')^2}\right) = O\left(\sum_{r \leq r' \leq T} \frac{1}{(r')^2}\right) = O(M^2 \log(MT)),
\]
and the result follows.

9.22. Lemma 9.22. Defining \(m, n, M\) as before, and supposing \(T' < t < T\),
\[
\int_T^{T+U} \frac{z(t) \varepsilon(t)(n^M)\min(m, n)^2}{\min(m, n)^2} dt = \frac{U}{(\log T)^2} \sum_{r \leq r' \leq T} \frac{1}{(r')^2} + O(U^2/T) + O(U^2/T) + O(T^3/k)
\]
(9.22.1)

if \(n < m\). If \(m < n\), the first term on the right-hand side is to be omitted.

The left-hand side is
\[
e^{-\frac{t}{2|\log t|}} \sum_{\mu \leq \alpha \leq \nu} \sum_{r \leq r' \leq \alpha} \frac{1}{r^2} \int_T^{T+U} \frac{n^M}{\min(m, n)^2} dt.
\]

The integral is of the form considered in § 4.6, with
\[
F(t) = \log \left(\frac{t}{\alpha^2}\right) e^{-\frac{t}{2|\log t|}}
\]

Hence by (4.6.5), with \(\lambda = (T + U)^{-1}\), \(\lambda' = (T' + U)^{-1}\), it is equal to
\[
2\pi x \exp(1 - x) + O(T) + \Theta\left(\frac{1}{\log(T + U)/\alpha}\right)
\]
(9.22.2)

with the leading term present only when \(T < \alpha < T + U\). We therefore obtain a main term
\[
2\pi x \frac{M^2}{\alpha^2} \sum_{r \leq r' \leq T} \frac{1}{(r')^2} + O(x M^2 / \alpha^2)
\]
(9.22.3)
where \(\mu\) and \(\nu\) also satisfy
\[
\frac{\lambda^2}{\alpha^2} / \mu \leq \nu \leq \frac{\lambda^2}{\alpha^2} / \mu.
\]

The double sum is clearly zero unless \(\alpha \leq m\), as we now suppose. The \(\psi\)-summation runs over the range \(1 \leq \nu \leq \mu\), where \(\nu = \mu^2 / m^2\) and \(\mu = \min(\psi^2 / m, 1)\), and \(\psi\) runs over \(1 \leq \nu \leq m\). The inner sum is therefore \((\psi^2 / m)^2 + O(\nu)\) if \(n = \psi m^2\), and \(O(\nu)\) otherwise. The error term \(O(\nu)\) contributes \(O((\psi^2 / m)^2) = O(MT^2)\) in (9.22.1). On writing \(\psi = \nu / \sqrt{m}\) we are left with
\[
2\pi \frac{M^2}{\alpha^2} \sum_{\alpha \leq r' \leq T} \frac{1}{(r')^2}.
\]
Let \(\psi = \nu / \sqrt{m}\). Then \(\psi = \nu / \sqrt{m}\) unless \(\nu < \sqrt{m}/\mu\). Hence the error on
replacing $\nu_2$ by $\nu_1$ is

$$
O\left(\frac{m}{n} \sum_{\nu \leq \nu_1 < \nu_2} \frac{\nu^2 - \nu_1 + 1}{\nu - \nu_1} \right) - O\left(\frac{m}{n} \sum_{\nu \leq \nu_1 < \nu_2} \frac{\nu^2 - \nu}{\nu - \nu_1} \right) - O\left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{\nu^2 - \nu_1 + 1}{\nu - \nu_1} \right) + O\left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{\nu^2 - \nu}{\nu - \nu_1} \right)
$$

$$
- O\left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{\nu^2 - \nu}{\nu - \nu_1} \right) - O\left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{\nu^2 - \nu}{\nu - \nu_1} \right)
$$

Finally there remains

$$
2\pi \left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{\nu^2 - \nu_1 + 1}{\nu - \nu_1} \right) - \frac{U}{(mn)^2} \sum_{\nu_1 < \nu < \nu_2 < \nu} \frac{1}{\nu - \nu_1}
$$

Now consider the $O$-terms arising from (9.22.3). The term $O(T^0)$ gives

$$
O\left(\frac{m}{n} \sum_{\nu_1 < \nu_2 < \nu} \frac{1}{\nu - \nu_1} \right) = O(T^{1/4}) = O(T^{1/4}).
$$

Next,

$$
\frac{1}{\nu - \nu_1} \sum_{\nu_1 < \nu_2 < \nu} \frac{1}{\nu - \nu_1} \min\left(\frac{1}{\log(2\pi \nu (\nu T)^{1/2})}, \frac{1}{\nu - \nu_1}ight)
$$

Suppose, for example, that $n < m$. Then the terms with $r < 3\pi n/m$ or $r > 3\pi n/m$ are

$$
O\left(\frac{1}{\nu - \nu_1} \right) = O(T^{1/4}).
$$

In the other terms, let $r = [n^2 m] - r'$. We obtain

$$
O\left(\frac{1}{\nu - \nu_1} \sum_{[n^2 m] < r'} \left(\frac{1}{r} \right) \min\left(\frac{1}{\log(2\pi \nu (\nu T)^{1/2})}, \frac{1}{\nu - \nu_1}ight) \right)
$$

omitting the terms $r' = -1, 0, 1$; and these are $O(T^{1/4})$.

A similar argument applies in the other case.

---

9.22. Lemma 9.23. Let $(m, n) = 1$ with $m, n < X < T^1$, if $T^{1/2} < U < T$, then

$$
\int_{T^{1/2}}^{U} \frac{Z(\nu^2)}{\nu^2} d\nu = -\frac{U}{(mn)^2} \log T^{2/3mn} + O(U T^{1-1/3} \log T).
$$

Let $Z(\nu) = \log(\nu^2 + \nu + 1)$. Then

$$
\int_{T^{1/2}}^{U} \left[ x_1(n) + x_2(n) \right]^2 \frac{n}{m} U^2 d\nu
$$

$$
- \int_{T^{1/2}}^{U} Z(\nu)^2 \frac{n}{m} U^2 d\nu + O\left(\int_{T^{1/2}}^{U} |Z(\nu)| d\nu\right) + O\left(\int_{T^{1/2}}^{U} |x_1(n)| d\nu\right)
$$

We have

$$
\int_{T^{1/2}}^{U} \ln(x^2) d\nu = O(U^{1/2} T) + O(T^{1/2} \log T) + O(U^{1/2} T^{1/2})
$$

by (9.30.7), and

$$
\int_{T^{1/2}}^{U} |Z(\nu)| d\nu = O(U^{1/2} T) + O(T^{1/2} \log T) = O(U^{1/2} T)
$$

by Theorem 7.4. Hence

$$
\int_{T^{1/2}}^{U} |Z(\nu)| d\nu = O\left((U^{1/2} T) \log T\right)
$$

by Cauchy's inequality. It follows that

$$
\int_{T^{1/2}}^{U} \frac{Z(\nu)^2}{\nu^2} d\nu
$$

$$
- \int_{T^{1/2}}^{U} \left[ x_1(n) + x_2(n) \right]^2 + 2 x_1(n) x_2(n) \frac{n}{m} U^2 d\nu + O(U^{1/2} T^{1-1/3} \log T).
$$

By Lemmas 9.21 and 9.22 the main integral on the right is

$$
\frac{U}{(mn)^2} \left(\sum_{r < \nu < \nu_1} \frac{1}{\nu - \nu_1} + \sum_{r < \nu_2 < \nu_1} \frac{1}{\nu - \nu_1} \right) + O(T^{1/4} \log(\log X)) + O(T^{1/4}) + O(U^{1/2} T^{1-1/3} \log T) + O(U^{1/2} T^{1/2})
$$
whether \( n \leq m \) or not. The result then follows, since

\[
\sum_{r \in \mathbb{C}^+} \frac{1}{2} \sum_{r \in \mathbb{C}^+} \frac{1}{r} = \log \frac{1}{m^n} + 2\gamma + O\left(\frac{X}{T}\right),
\]

and since the error terms \( O(T\log(XT)), O(XT^{1/2}), O(U^{1/2}), O(T^{1/2}) \) and \( O(U^{1/2}T^{-1/2}) \) are all \( O(U^{1/2}T^{-1}\log T) \).


Consider the integral

\[
I = \int \left( \frac{1}{(n+1)(n+1)} \right) dt = \int B(n) \frac{1}{(n+1)^2} dt,
\]

where

\[
\phi(n) = \sum_{r \in \mathbb{C}^+} \frac{n}{r}.
\]

and

\[
\delta_i = \sum_{r \in \mathbb{C}^+} \mu(n) \frac{n}{r} \phi(n) \phi(r).
\]

Clearly

\[
|\delta_i| \leq \frac{1}{\phi(n)}
\]

for all values of \( n \). Now

\[
I = \sum_{r \in \mathbb{C}^+} \sum_{r \in \mathbb{C}^+} \delta_i \phi(n) \phi(r)
\]

where \( m = q(n,r) \), \( n = q(n,r) \). Using Lemma 9.23, the main term contributes to this

\[
\sum_{r \in \mathbb{C}^+} \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n - \frac{1}{2} \sum_{r \in \mathbb{C}^+} \int_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

\[
= \frac{1}{2} \int_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n - \frac{1}{2} \sum_{r \in \mathbb{C}^+} \int_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

For a fixed \( q < X \),

\[
\sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \left( \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n \right) - \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Now

\[
\sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \left( \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n \right) - \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Hence the second factor on the right is

\[
\sum_{r \in \mathbb{C}^+} \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \sum_{r \in \mathbb{C}^+} \phi(n) \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Put \( r = 1 \). Then \( \rho = \rho(1) \), i.e., \( \rho(1) \). Hence we get

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \sum_{r \in \mathbb{C}^+} \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

The \( \rho \)-sum is 0 unless \( l = 1 \), when it is 1. Hence we get

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Hence

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

and

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Let \( \phi(n) \) be defined by

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

so that

\[
\phi(n) = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Then

\[
\phi(n) = \sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

and hence

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

Hence

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]

and

\[
\sum_{r \in \mathbb{C}^+} \phi(n) \frac{1}{r} \frac{1}{n^{1/2}} \frac{1}{2} \log 2 \pi n
\]
Replacing $T$ by $T + U, T + 2U, \ldots$ and adding, $O(\log T)$ terms, we obtain
\[
\int \frac{1}{4} [N(s, T + U) - N(s, T)] \, ds = O(T).
\]
Replacing $T$ by $\frac{3}{2} T, \frac{5}{2} T, \ldots$ and adding, the theorem follows.

It also follows that, if \( \frac{1}{T} < s < 1 \),
\[
N(s, T) \leq \frac{3}{2} \int \frac{1}{4} [N(s, T)] \, ds
\]
Lastly, if \( \phi(t) \) is positive and increases to infinity with \( t \), all but an infinitesimal proportion of the zeros of \( \xi(s) \) lie in the upper half-plane in the region
\[
|s - \frac{1}{2}| > \frac{\phi(t)}{\log t}
\]
The curved boundary of the region
\[
s = \frac{3}{4}, \quad T^4 < t < T
\]
lies to the right of
\[
s = s_1 = \frac{3}{4} + \frac{\phi(t)}{\log t}
\]
and
\[
N(s, T) = O\left( \frac{T}{(s_1 - \frac{3}{4})} \right) = O\left( \frac{T \log T}{(s_1 - \frac{3}{4})} \right) = o(T \log T).
\]
Hence the number of zeros outside the region specified is \( o(T \log T) \), and the result follows.

**NOTES FOR CHAPTER 9**

### 9.25. The mean value of \( \zeta(s) \) has been investigated by Selberg (5). One has
\[
\int \frac{1}{2} [\zeta(s)]^2 \, ds - \frac{2(2\pi)^{3/2}}{k(2\pi)^{1/2}} T (\log T)^4
\]
for every positive integer \( k \). Selberg's earlier conditional treatment (6) is discussed in §§ 4.20-4.26, the key features used in (8) so dealing with the critical line being the estimate given in Theorem 9.18(C). Selberg (5) also gave an unconditional proof of Theorem 14.19, which had previously been established on the Riemann hypothesis by Littlewood.
These results have been investigated further by Fujiwara [1], [2] and Ghosh [1], [7] who give results which are uniform in $k$.

It follows in particular from Fujiwara [1] that

$$
\int_0^T |S(t + h) - S(t)|^\delta dt < \pi^{-2} T \log(3 + h \log T) + O\left(T \left(\log(3 + h \log T)^{1/2}\right)\right)
$$

(9.25.2)

and

$$
\int_0^T |S(t + h) - S(t)|^\delta dt \leq T \left(A k^1 \log(3 + h \log T)^{1/2}\right)^\delta
$$

(9.25.3)

uniformly for $0 < h < \frac{1}{2} T$. One may readily deduce that

$$
N_j(T) \leq N(T) e^{-\delta \sqrt{\lambda}},
$$

where $N_j(T)$ denotes the number of zeros $\beta + i \gamma$ of multiplicity exactly $j$, in the range $0 < \gamma < T$. Moreover one finds that

$$
\# \{n: 0 < \gamma_n < T, \gamma_{n+1} - \gamma_n > \lambda / \log T\} \leq N(T) \exp\left[-A k^1 \log(\lambda)^{-1}\right],
$$

uniformly for $\lambda > 2$, whence, in particular,

$$
\sum_{0 < \gamma_n < T} (\gamma_{n+1} - \gamma_n)^\delta \leq \frac{N(T)}{(\log T)^{1/2}}
$$

(9.25.4)

for any fixed $\delta > 0$. Fujiwara [2] also states that there exist constants $\lambda > 1$ and $\mu < 1$ such that

$$
\gamma_{n+1} - \gamma_n \geq \lambda / \log \gamma_n
$$

(9.25.5)

and

$$
\gamma_{n+1} - \gamma_n \leq \mu / \log \gamma_n
$$

(9.25.6)

each hold for a positive proportion of $n$ (i.e. the number of $n$ for which $0 < \gamma_n < T$ is at least $AN(T)$ if $T > T_0$). Note that $2 \pi / \log \gamma_n$ is the average spacing between zeros. The possibility of results such as (9.25.6) and (9.25.6) was first observed by Selberg [1].

9.26. Since the deduction of the results (9.25.5) and (9.25.6) is not obvious, we give a sketch. If $M$ is a sufficiently large integer constant,

$$
\int_0^T |S(t + h) - S(t)|^\delta dt \geq T
$$

and

$$
\int_0^T |S(t + h) - S(t)|^\delta dt \leq T
$$

uniformly for $2 \pi M / \log T < h < 2 \pi M / \log T$.

By Hölder's inequality we have

$$
\int_0^T |S(t + h) - S(t)|^\delta dt \leq \left(\int_0^T |S(t + h) - S(t)| dt\right)^{1/2} \left(\int_0^T |S(t + h) - S(t)|^\delta dt\right)^{1/2}
$$

(9.25.7)

so that

$$
\int_0^T |S(t + h) - S(t)|^\delta dt \geq T.
$$

We now observe that

$$
S(t + h) - S(t) = N(t + h) - N(t) - \frac{h \log T}{2 \pi} + O\left(\frac{1}{\log T}\right)
$$

for $T < t < 2T$, whence

$$
\int_0^T |N(t + h) - N(t)| \frac{h \log T}{2 \pi} dt \geq T.
$$

We proceed to write $h = 2 \pi M / \log T$ and

$$
\delta(t, \lambda) = N\left(\frac{2 \pi M}{\log T}\right) - N(t) - \lambda
$$
so that
\[ N(t) - N(t_{0}) - \frac{A \log T}{2x} = \sum_{n=1}^{M-1} h \left( \frac{t + 2 \pi n}{\log T} \right) \]

Thus
\[ T \leq \sum_{n=1}^{M-1} \int_{t+2\pi n/\log T}^{t+2\pi (n+1)/\log T} |d(\xi, \lambda)| dt \]
\[ = M \int_{t}^{t+2\pi M/\log T} |d(\xi, \lambda)| dt + O(1), \]
and hence
\[ \int_{t}^{t+2\pi M/\log T} |d(\xi, \lambda)| dt \geq T \quad (9.26.1) \]
uniformly for \(1 < \lambda < 2\), since \(M\) is constant.

Now, if \(I\) is the subset of \([T, 2T]\) on which \(N\left( t + \frac{2 \pi n}{\log T} \right) = N(t)\), then
\[ |d(\xi, \lambda)| \leq \begin{cases} 0 & (t \in I), \\ \frac{2 \pi n}{\log T} & (t \notin I) \end{cases} \]
so that (9.36.3) yields
\[ T \leq \int_{I} |d(\xi, \lambda)| dt + \frac{(2 \pi n/2 \log T) + 2m(I)}{2} \]
where \(m(I)\) is the measure of \(I\). However
\[ \int_{I} |d(\xi, \lambda)| dt = O\left( \frac{T}{\log T} \right) \]
whence \(m(I) \gg T\), if \(I > 1\) is chosen sufficiently close to 1. Thus, if
\[ S = \left\{ n: T \leq \gamma_{n} \leq 2T, \gamma_{n+1} - \gamma_{n} \geq \frac{2 \pi n}{\log T} \right\} \]
then
\[ T \cdot m(I) \leq \sum_{n \in S} (\gamma_{n+1} - \gamma_{n}) + O(1). \]

9.36. General Distribution of Zeros

so that
\[ T^{2} \leq \left( \sum_{n \in S} (\gamma_{n+1} - \gamma_{n}) \right)^{2} \leq \frac{2 S}{\log T} \leq \frac{T}{2x} \]
by (9.35.4) with \(k = 2\). It follows that
\[ \# S \gg N(T) \]
proving that (9.35.5) holds for a positive proportion of \(n\).

Now suppose that \(\mu\) is a constant in the range \(0 < \mu < 1\), and put
\[ U = \{ n: T \leq \gamma_{n} \leq 2T \}, \]
and
\[ V = \{ n \in U: \gamma_{n+1} - \gamma_{n} \leq \frac{2 \pi n}{\log T} \}, \]
whence \(\# U = \frac{T}{2x} \log T + O(T)\). Then
\[ T = \sum_{n \in U} (\gamma_{n+1} - \gamma_{n}) + O(1) \]
\[ \geq \sum_{n \in V} (\gamma_{n+1} - \gamma_{n}) + O(1) \]
\[ \geq \frac{2 \pi n}{\log T} (\# U - \# V) + \frac{2 \pi n}{\log T} S + O(1) \]
\[ = \frac{2 \pi n}{\log T} \left( \frac{T}{2x} \log T - \# V \right) + \frac{2 \pi n}{\log T} \# S + O\left( \frac{T}{\log T} \right). \]
If the implied constant in (9.36.3) is \(n\), it follows that \(\# V \gg N(T)\), on taking \(\mu = 1 - r, 0 < r < n(\lambda - 1)/(1 - n)\). Thus (9.35.6) also holds for a positive proportion of \(n\).

9.37. Gosh (1) was able to sharpen the result of Selberg mentioned at the end of 9.10, to show that \(S(\delta)\) has at least
\[ T(\log T)^{-A/2} \exp \left( -\frac{A \log T}{(\log(\log T))^{\delta}} \right) \]
sign changes in the range \(0 < t < T\), for any positive \(\delta\), and \(A = A(\delta), T = T(\delta)\). He also proved (Gosh (3)) that the asymptotic formula (9.25.1) holds for any positive real \(k\), with the constant on the right hand
side replaced by $T(2k + 1)/(2 + 1)(2 + 1/3)^{3k}$. Moreover he showed (Ghosh [2]) that

$$f(t) = \frac{2\pi}{\sqrt{(\log\log t)}}$$

say, has a limiting distribution

$$P(\varepsilon) = 2\pi \int_0^\infty e^{-\varepsilon t} dt,$$

in the sense that, for any $\varepsilon > 0$, the measure of the set of $t \in [0, T]$ for which $f(t) < \varepsilon$, is asymptotically $TP(\varepsilon)$. (A minor error in Ghosh's statement of the result has been corrected here.)

9.28. A great deal of work has been done on the 'zero-density estimates' of §§9.15–19, using an idea which originates with Halász [1]. However it is not possible to combine this with the method of §§9.16, based on Littlewood's formula (9.9.1). Instead one argues as follows (Montgomery [1, Chapter 12]). Let

$$M_{\rho}(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

so that $a_n = 0$ for $2 \leq n \leq X$ if $\{\rho\} = 0$, where $\rho = \beta + iy$ and $\beta > 1/2$ then we have

$$e^{-\gamma T + \sum_{n \leq T} \frac{a_n}{n} e^{-\gamma n}} = \sum_{n \leq T} a_n n^{-\gamma}$$

$$= \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} M(x + \rho) (x + \rho)^{-\gamma} ds,$$

by the lemma of §7.9. On moving the line of integration to $R = \frac{1}{2} - \beta$ this yields

$$M_{\rho}(1) (1 - \beta) Y^{-1 + \beta} +$$

$$+ \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} M(x + \rho) (x + \rho)^{1 - \beta} - x^{-\beta - (t - i\gamma)} dt,$$

since the pole of $\Gamma(s)$ at $s = 0$ is cancelled by the zero of $(s + \rho)$. If we now assume that $T < 1 \leq T$, and that $\log T \leq \log X$, $\log Y < \log T$,

$$M_{\rho}(1) Y^{-1 + \beta} \rightarrow 0$$

then $e^{-\gamma T} \rightarrow 1$ and

$$M_{\rho}(1) (1 - \beta) Y^{-1 + \beta} = o(1),$$

whence either

$$\sum_{n \leq T} a_n n^{-\gamma} \rightarrow 1$$

or

$$\int_{\rho - i\infty}^{\rho + i\infty} |M_{\rho}(x + \rho) (x + \rho)^{-\gamma} ds \rightarrow Y^{1 + \beta}.$$

In the latter case one has

$$M_{\rho}(1) (1 - \beta) Y^{-1 + \beta} \Rightarrow (x - 1) Y^{1/2},$$

for some $t_x$ in the range $|t_x - i\gamma| \leq \log T$. The problem therefore reduces to that of counting discrete points at which one of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-\gamma} M_{\rho}(n)$, and $(\rho)$ is large. In practice it is more convenient to take finite Dirichlet polynomials approximating to these.

The methods given in §§9.17–19 correspond to the use of a mean-value bound. Thus Montgomery [1, Chapter 7] showed that

$$\sum_{r=1}^{T} \sum_{n=1}^{\infty} a_n n^{-\gamma} \cdot (T + N)(\log N)^{\gamma} \sum_{n=1}^{N} |a_n|^{1/2} \gamma (2.31)$$

for any points $n_s$ satisfying

$$R(n_s) \geq \gamma, \quad |\{n_s\}| < T, \quad \{n_s\} \geq 1.$$

(2.29.2)

and any complex $a_n$. Theorems 9.17, 9.18, 9.19(A), and 9.19(B) may all be recovered from this (except possibly for worse powers of $\log T$). However one may also use Halász's lemma. One simple form of this (Montgomery [1, Theorem 2.12]) gives

$$\sum_{r=1}^{T} \sum_{n=1}^{\infty} a_n n^{-\gamma} \cdot \log (N + RT) \sum_{n=1}^{N} |a_n|^{1/2} \gamma (2.32.3)$$

for any points $n_s$ satisfying (2.32.3). Under suitable circumstances this implies a sharper bound for $R$ than does (2.32.3). Under the Lindelöf hypothesis one may replace the term $RT\gamma (2.32.3)$ by $RT\gamma^2(1/2)$, which is superior.
Conjectured:
\[ \sum_{n \leq T^{\frac{1}{2}}} \sum_{a \leq T} a_n^{\sigma - 1} \ll (N + RT) \sum_{n \leq T} |a_n|^{\sigma - \frac{1}{2}} \]
for points \( \sigma \), satisfying (9.29.2). Using the Halasz lemma with the Lindelof hypothesis one obtains
\[ N(\sigma, T) \ll T^{\frac{1}{2} + \varepsilon} \,
\, T < 1, \quad \varepsilon > 0, \quad \sigma < 1, \quad (9.29.4) \]
(Halasz and Turan [1], Montgomery [1; Theorem 12.3]). If the Large Values Conjecture is true then the Lindelof hypothesis gives the wider range \( \frac{1}{2} + \varepsilon < \sigma < 1 \) for (9.29.4).

9.29. The picture for unconditional estimates is more complex. At present it seems that the Halasz method is only useful for \( \sigma > \frac{3}{4} \). Thus Ivingham's result, Theorem 9.19(II), is still the best known for \( \frac{1}{2} < \sigma < \frac{3}{4} \).

Using (9.29.3), Montgomery [1; Theorem 12.1] showed that
\[ N(\sigma, T) \ll T^{\frac{1}{4} - \sigma} \,(\log T)^{\varepsilon} \quad \left( \varepsilon < \frac{1}{4} \right), \]
which is superior to Theorem 9.19(B). This was improved by Huxley [1] to give
\[ N(\sigma, T) \ll T^{\frac{1}{4} - \sigma} \,(\log T)^{\varepsilon} \quad \left( \varepsilon < \frac{1}{4} \right). \quad (9.29.1) \]

Huxley used the Halasz lemma in the form
\[ R \ll \left( \sum_{n \leq T} |a_n|^{\sigma - 1} \right)^{\lambda} (\log T)^{\varepsilon}, \]
for points \( \sigma \), satisfying (9.29.2) and the condition
\[ \sum_{n \leq T} |a_n|^{\sigma - 1} \delta > \frac{1}{2}. \]
In conjunction with Theorem 9.19(B), Huxley's result yields
\[ N(\sigma, T) \ll T^{\frac{1}{8} + \varepsilon} \,(\log T)^{\varepsilon} \quad \left( \varepsilon < \frac{1}{4} \right). \]
(c.f. (9.18.3)). A considerable number of other estimates have been given, for which the interested reader is referred to Ivić [8; Chapter 11]. We mention only a few of the most significant. Ivić [3] showed that
\[ N(\sigma, T) \ll \left\{ \begin{array}{ll}
T^{\frac{1}{2} - \frac{1}{2} \sigma \delta} & \left( \frac{1}{2} < \sigma < \frac{3}{4} \right) \\
T^{\frac{1}{4} - \frac{1}{4} \sigma \delta} & \left( \frac{3}{4} < \sigma < 1 \right)
\end{array} \right. \]
which supersedes Huxley's result (9.29.1) throughout the range \( \frac{1}{2} < \sigma < 1 \). Jutila [11] gave a more powerful, but more complicated, result, which has a similar effect. His bounds also imply the 'Density hypothesis' \( N(\sigma, T) \ll T^{1 - \alpha \sigma} \), for \( \frac{1}{2} < \sigma < 1 \). Heath-Brown [8] improved this by giving
\[ N(\sigma, T) \ll T^{1 - \frac{1}{2} \sigma \delta} \,(\log T)^{\varepsilon} \quad \left( \varepsilon < \frac{1}{4} \right). \]
When \( \sigma \) is very close to \( 1 \) one can use the Vinogradov-Korobov exponential sum estimates, as described in Chapter 6. These lead to
\[ N(\sigma, T) \ll T^{\frac{1}{2} - \alpha \sigma \delta} \,(\log T)^{\varepsilon} \]
for suitable numerical constants \( \alpha \) and \( \delta \), (see Montgomery [1; Corollary 12.5]), who gives \( \alpha = 1.324 \), after correction of a numerical error.

Seiberg's estimate given in Theorem 9.19(C) has been improved by Jutila [7] to give
\[ N(\sigma, T) \ll T^{\frac{1}{8} - \sigma \delta} \,(\log T)^{\varepsilon} \]
uniformly for \( \frac{1}{2} < \sigma < 1 \), for any fixed \( \delta > 0 \).

9.30. Of course Theorem 9.24 is an immediate consequence of Theorem 10.19(C), but the proof is a little easier. The coefficients \( \mu \), used in §9.24 are essentially
\[ \mu(r)^{-1} \log X/r \log X, \]
and indeed a more careful analysis yields
\[ \int_{|r| < \log T} \left| \sum_{n \leq T} \mu(n) \right|^2 \frac{\left\{ \log X \right\} \log X \log T}{r \log X} dr = \sum_{n \leq T} \mu(n) \frac{\log X}{r \log X} \log T \]
Here one can take \( X < T^{\frac{1}{4}} \) using fairly standard techniques, or \( X < T^{1/3} \) by employing estimates for Kloosterman sums (see Balasubramanian, Conrey and Heath-Brown [1]). The latter result yields (9.24.1) with the implied constant 0.00845.
0.1. General discussion. The memoir in which Riemann first considered the zeta-function has become famous for the number of ideas it contains which have since proved fruitful, and it is by no means certain that these are ever now exhausted. The analysis which precedes his observations on the zeros is particularly interesting. He obtains, as in § 2.6, the formula
\[
\Gamma(s)\zeta(s) = \frac{1}{s(1-s)} + \frac{1}{\pi} \int_0^\infty \frac{\psi(x)}{x^{s+1}} \sin(x-1+b) \, dx,
\]
where
\[\psi(x) = \sum \frac{\cos(x)}{x^{a(1-\epsilon)}}.
\]
Multiplying by \(\epsilon^\theta\), and putting \(s = \frac{1}{2} + it\), we obtain
\[
\Xi(t) = \frac{1}{2} (\theta + 1) \int \frac{\psi(x)}{2x} \cos(\frac{1}{2} \log x) \, dx.
\]
Integrating by parts, and using the relation
\[\psi(x)\cos(\frac{1}{2} \log x) = \frac{1}{2} \cos(\frac{1}{2} \log x),
\]
which follows at once from (2.6.2), we obtain
\[
\Xi(t) = 4 \int \frac{1}{2x} \left( \frac{\psi(x)}{2x} \right) \cos(\frac{1}{2} \log x) \, dx.
\]
Riemann then observes:

'Diese Funktion ist für alle endlichen Werte von \(t\) endlich, und lässt sich nach Potenzen von \(t\) in eine sehr schnell divergierende Reihe entwickeln. Da für einen Wert von \(s\), dessen reelle Bestimmtheit größer als 1 ist, \(\log(\zeta) = - \sum \log(1-p^n)\)
enden bleiben, und von den Logarithmen der übrigen Funktionen von \(s\) dieselbe
gilt, so dass die Funktion \(\Xi(t)\) nur verschiedene, wenn der imaginäre Teil von \(t\) zwischen \(\frac{1}{2} + \frac{1}{2} \pi\) liegt. Die Anzahl der Wurzeln von \(\Xi(t)\) = 0, deren reeller
Teil zwischen 0 und \(T\) liegt, ist etwa
\[
\frac{T}{\pi} \log T.
\]
denn dass Integral \(\int \log(\zeta)\) positive um den Inbegriff der Wurzeln von \(t\)
estrecken, deren imaginären Teil zwischen \(\frac{1}{2}\) und \(-\frac{1}{2}\), und dessen reeller Teil zwischen \(0\) und \(T\) liegt, so dass auf einem Umkreis von der Ordnung \(|\zeta| = 1/T\) gleich \(|T\log(2T)| - \frac{T}{2}\), dieses Integral aber ist gleich der Anzahl der in diesem Gebiete liegenden Wurzeln von \(\Xi(t) = 0\), multipliziert mit zwei. Man findet
also in der That etwa so viele Wurzeln innerhalb dieser Grenze, und es ist
sehr wahrscheinlich, dass alle Wurzeln reelle sind.'

This statement, that all the zeros of \(\Xi(t)\) are real, is the famous
'Riemann hypothesis', which remains unproved to this day. The memoir
opens:

'Hierzu wäre allerdings eine strenge Beweis aufzuschreiben; ich habe indessen die
Anforderung derselben nach einigen Säuglingen vergeblich versuchen. Vorläufig
sei eine Versuchung vorgenommen, da sie mir wohltut, und dass es meiner
Untersuchung (i.e. the explicit formula for \(\psi(x)\)) vorzustellen scheint.'

In the approximate formula for \(\psi(t)\), Riemann's \(1/T\) may be a
mistake for \(\log T\); for, since \(\psi(t)\) has an infinity of discontinuities at
least equal to 1, the remainder cannot tend to zero. With this correction,
Riemann's first statement is Theorem 9.4, which was proved by von
Mangoldt many years later.

Riemann's second statement, on the real zeros of \(\Xi(t)\), is more obscure,
and its exact meaning cannot now be known. It is, however, possible
that anyone encountering the subject for the first time might argue as
follows. We can write (10.1.2) in the form
\[
\Xi(t) = 2 \int \Phi(w) \cos(\frac{1}{2} \log x) \, dw,
\]
where
\[\Phi(w) = 2 \sum_{n=1}^{\infty} \left(2 \sinh \pi w \right) e^{-\pi n x} \cos(\pi n x).
\]
This series converges very rapidly, and one might suppose that an
approximation to the truth could be obtained by replacing it by its first
term; or perhaps better by
\[\Phi(w) = 2 \sinh \pi w x \cos(\pi n x),
\]
since this, like \(\Phi(w)\), is an even function of \(w\), which is asymptotically
equivalent to \(\Phi(w)\). We should thus replace \(\Xi(t)\) by
\[
\Xi(t) = 2 \int \frac{2 \sinh \pi w x \cos(\pi n x)}{x} \cos(\frac{1}{2} \log x) \, dx.
\]
The asymptotic behaviour of \(\Xi(t)\) can be found by the usual method
of steepest descents. To avoid the calculation we shall quote known
Bessel-function formulas. We have:
\[
K_j(x) = \int_0^\infty e^{-t} \cos(\pi n x) \, dt,
\]
and
\[
\Xi(t) = -e^{-\pi T} \Phi(T) + K_j(e^{-\pi T}).
\]
For fixed \(x\), as \(n \to \infty\),
\[
L(x) \sim (\log x)^{1/(e+1)}.
\]
\[\text{Watson, Theory of Bessel Functions, 6.22 (1).} \]
Hence
\[
L_{\pm 1}(2\pi) \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1 + \frac{1}{2}b}} e^{-\frac{1}{2}b},
\]
\[
R_{\pm 1}(2\pi) \sim \frac{1}{\sqrt{2\pi}} (1 + \frac{1}{2}b) - O(b),
\]
\[
K_{\pm 1}(2\pi) \sim \frac{1}{\sqrt{2\pi}} \cos \frac{1}{2} \log \frac{1}{\sqrt{2\pi} + \frac{1}{2}b},
\]
\[
\sim \frac{1}{\sqrt{2\pi}} \exp \left( \frac{1}{2} \log \frac{1}{\sqrt{2\pi} + \frac{1}{2}b} \right).
\]

Hence
\[
\Xi(t) \sim 2\pi L \cos \left( \frac{1}{\sqrt{2\pi} + \frac{1}{2}b} \right).
\]
The right-hand side has zeros at
\[
\pm \log \frac{1}{\sqrt{2\pi} + \frac{1}{2}b},
\]
and the number of these in the interval \((0, T)\) is
\[
T \log \frac{1}{\sqrt{2\pi} + \frac{1}{2}b} + O(1).
\]
The similarity to the formula for \(N(t)\) is indeed striking.

However, if we try to work on this suggestion, difficulties do not appear.

We write
\[
\Xi(t) - \Xi'(t) = \int \phi(u) - \Phi(u) e^{\mu u} du.
\]

To show that this is small compared with \(\Xi(t)\) we should want to move
the line of integration into the upper half plane, at least as far as
\(\Re(u) = \frac{1}{2}\); and this is just where the series for \(\Phi(u)\) comes to converge.

Actually
\[
\Xi(t) > A t e^{-\frac{1}{2}b} \left( 1 + t \right),
\]
and \((\Xi + i)\) is unbounded, so that the suggestion that \(\Xi(t)\) is an
approximation to \(\Xi(t)\) is false, at any rate if it is taken in the most
obvious sense.

10.2. Although every attempt to prove the Riemann hypothesis, that
all the complex zeros of \(\zeta\) lie on \(\Re(s) = \frac{1}{2}\), has failed, it is known that
\(\zeta\) has an infinity of zeros on \(\Re(s) = 1\). This was first proved by Hardy
in 1916. We shall give here a number of different proofs of this theorem.

First method. We have
\[
\Xi(t) = -\frac{1}{2} t e^{-\frac{1}{2}b} \Pi(1 + \frac{1}{2}b) \zeta(1 + i),
\]
where \(\Xi(t)\) is an even integral function of \(t\), and is real for real \(t\). A zero
\ \text{Hardy (1)}.
for all $z < \frac{1}{2}$ and $T > 7$. Hence, making $z \rightarrow \frac{1}{2}$,
\[
\int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT < L.
\]
Hence the integral
\[
\int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT
\]
is convergent. The integral on the left of (10.2.2) is therefore uniformly convergent with respect to $n$ for $0 \leq n \leq 1$, and it follows that
\[
\sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT = \frac{(-1)^{n} \cos \frac{1}{2} \pi T}{2^{n}}
\]
for every $n$.

This, however, is impossible; for, taking $n$ odd, the right-hand side is negative, and hence
\[
\int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT < - \int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT < K T^{m},
\]
where $K$ is independent of $n$. But by hypothesis there is a positive $m$ such that $\sum_{n} T^{n} \zeta(1/2 + i \sigma) \geq m$ for $2 T^{n} \ll T^{2} + 1$. Hence
\[
\int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT \leq \int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT \geq m T^{m}. \quad \text{(10.2.3)}
\]
Hence
\[
m T^{m} < K.
\]
which is false for sufficiently large $n$. This proves the theorem.

10.3. A variant of the above proof depends on the following theorem of Fejer:

Let $n$ be any positive integer. Then the number of changes in sign in the interval $(0, a)$ of a continuous function $f(t)$ is not less than the number of changes in sign of the sequence
\[
f(0), f(a - 0)dt, \ldots, f(a)dt. \quad \text{(10.3.1)}
\]

We deduce this from the following theorem of Paley:

Let $n$ be any positive integer. Then the number of changes in sign in the interval $(0, a)$ of a continuous function $f(t)$ is not less than the number of changes in sign of the sequence
\[
f(0), f(a - 0)dt, \ldots, f(a)dt. \quad \text{(10.3.1)}
\]

To prove Fejer's theorem, suppose first that $n = 1$. Consider the curve $y = f(x)$. Now $f(0) = 0$, and, if $f(a)$ and $f(b)$ have opposite signs, $y$ is positive decreasing or negative increasing at $x = a$. Hence $f(x)$ has at least one zero.

Now assume the theorem for $n - 1$. Suppose that there are $k$ changes of sign in the sequence $f(a), f(b), \ldots, f(c)$. Then $f(x)$ has at least $k$ changes of sign. We have then to prove that

(i) if $f(a)$ and $f(b)$ have the same sign, $f(x)$ has at least $k + 1$ changes of sign,

(ii) if $f(a)$ and $f(b)$ have different signs, $f(x)$ has at least $k + 1$ changes of sign.

Each of these cases is easily verified by considering the curve $y = f(x)$. This proves Fejer's theorem.

To deduce Paley's theorem, we have
\[
f(x) = \frac{1}{(x - a)^{\frac{1}{2}}} \int_{a}^{x} f(t) dt,
\]
and hence
\[
f'(x) = \frac{1}{(x - a)^{\frac{1}{2}}} \int_{a}^{x} f(t) dt - \frac{1}{(x - a)^{\frac{1}{2}}} \int_{a}^{x} f(t) dt.
\]

We may therefore replace the sequence (10.3.2) by the sequence
\[
f(a), f(a - 0) dt, \ldots, f(a - 0) dt. \quad \text{(10.3.3)}
\]

Since the number of changes of sign of $f(0)$ is the same as the number of changes of sign of $f(a - 0)$, we can replace $f(0)$ by $f(a - 0)$. This proves Fejer's theorem.

To prove that there are an infinity of zeros of $f(x)$ on the critical line, we prove as before that
\[
\lim_{x \rightarrow \infty} \int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT = \frac{(-1)^{m} \cos \frac{1}{2} \pi}{2^{m}}
\]
Hence
\[
\int_{\frac{1}{2}}^{T} \sum_{n} \frac{\zeta(1/2 + i \sigma)}{\rho_{n}} \cos \frac{1}{2} \pi T dT
\]
has the same sign as \((-1)^n\) for \(n = 0, 1, \ldots, N\), if \(\sigma = \alpha(N)\) is large enough and \(\alpha = \alpha(N)\) is near enough to \(1/2\). Hence \(\Xi(t)\) has at least \(N\) changes of sign in \((0, a)\), and the result follows.†

10.4. Another method is based on Riemann's formula (10.1.2).

Putting \(\pi = \pi^n\) in (10.1.2), we have

\[
\Xi(t) = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi(n) \cos nt \, dn
\]

say. Then, by Fourier's integral theorem,

\[
\Phi(n) = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt,
\]

and hence also

\[
\Phi(n^2) = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt.
\]

Since \(\Phi(x)\) is regular for \(\Re(x) > 0\), \(\Phi(n)\) is regular for \(-\frac{1}{2} < \Im(n) < \frac{1}{2}\).

Let

\[
\Phi(n) = \varepsilon_1 + \varepsilon_1^2 n + \varepsilon_1^3 n^2 + \ldots \quad |\varepsilon_1| < \frac{1}{2}.
\]

Then

\[
(2n)! \varepsilon_1 = (-1)^n \Phi(n^2)(0) = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt.
\]

Suppose now that \(\Xi(t)\) is of one sign, say \(\Xi(t) > 0\), for \(t > T\). Then \(\varepsilon_1 > 0\) for \(n > n_0\), since

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt > \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt > (T + 1)^{n_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Xi(t) \cos nt \, dt.
\]

It follows that \(\Phi(n^2)\) increases steadily with \(n\) if \(n > n_0\). But in fact \(\Phi(n)\) and all its derivatives tend to 0 as \(n \to \infty\) along the imaginary axis, by the properties of \(\xi(x)\) obtained in §10.2. The theorem therefore follows again.

10.5. The above proofs of Hardy's theorem are all similar in that they depend on the consideration of 'moments' \(\int_0^a d\tau\). The following

† Selbke (D).

† Perly (B).

method† depends on a contrast between the asymptotic behaviour of the integrals

\[
\int_0^\infty \Xi(t) \, dt, \quad \int_0^\infty \Xi(t) \, dt,
\]

where \(\Xi(t)\) is the function defined in §4.17. If \(\Xi(t)\) were ultimately of one sign, these integrals would be ultimately equal, apart possibly from sign. But we shall see that in fact they behave quite differently.

Consider the integral

\[
\int_0^{\infty} \left| \xi(t) \right| \frac{1}{2 \pi} \Xi(t) \, dt,
\]

where the integrand is the function which reduces to \(\Xi(t)\) on \(\sigma = \frac{1}{2}\), taken round the rectangle with sides \(\sigma = \frac{1}{2}, \sigma = \frac{1}{2}, t = T, t = 2T\).

This integral is zero, by Cauchy's theorem. Now

\[
\int_0^{\infty} \left| \xi(t) \right| \frac{1}{2 \pi} \Xi(t) \, dt = \frac{1}{\pi} \int_0^\infty \xi(t) \Xi(t) \, dt.
\]

By (4.12.3)

\[
\xi(t) = 1 + \frac{1}{t} + \frac{1}{t} - \frac{1}{t} + \frac{1}{1 + t} + O\left(\frac{1}{t}^2\right).
\]

Hence, by (5.1.2) and (5.1.4),

\[
\xi(t) = O\left(\frac{1}{t}^2\right) = O\left(\frac{1}{t}^2\right) = O\left(\frac{1}{t}^2\right) = O\left(\frac{1}{t}^2\right).
\]

The integrals along the sides \(t = T, t = 2T\) are therefore \(O(T^{-1})\).

The integral along the right-hand side is

\[
\int_0^\infty \left| \xi(t) \right| \frac{1}{2 \pi} \Xi(t) \, dt = O(T^{-1}).
\]

The contribution of the \(O\)-term is

\[
\int_0^\infty 0 \, dt = O(T^2).
\]

The other term is a constant multiple of

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 x^{s-1} e^{-\pi x^2 n} \, dx.
\]

Now

\[
\int_0^1 x^{s-1} e^{-\pi x^2 n} \, dx = \frac{1}{2 \pi n^{1/2} \sqrt{\pi}} e^{-\pi n}.
\]

Hence, by Lemma 4.5, the integral in the above sum is \(O(T^2\log T)\), uniformly with respect to \(s\), so that the whole sum is also \(O(T^2\log T)\).

† See Landau, Vorlesung Uran., ii. 19-22.
Combining all these results, we obtain
\[
\int_{T}^{T'} |Z(t)| \, dt = O(T^2). \tag{10.6.1}
\]

On the other hand,
\[
\int_{T}^{T'} |Z(t)| \, dt = \int_{1}^{T'} \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt \geq \int_{1}^{T'} \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt.
\]

But
\[
\int_{1}^{T'} \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt = \int_{1}^{T'} \left( \zeta \left( \frac{1}{2} + it \right) \right) \, dt = \zeta(1) + \frac{1}{2} \int_{1}^{T'} \frac{1}{t} \, dt + \frac{1}{2} \int_{1}^{T'} \frac{1}{t} \, dt.
\]

Hence
\[
\int_{T}^{T'} |Z(t)| \, dt > AT. \tag{10.5.2}
\]

Hardy's theorem now follows from (10.6.1) and (10.5.2).

Another variant of this method is obtained by starting again from (10.5.2). Setting \( s \to 1^+ \), we obtain
\[
\int_{T}^{T'} |Z(t)| \, dt = \int_{1}^{T'} \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt = O(T^2 - \sum_{n \leq T/\log T} \exp(-\pi n \rho - \pi n \delta) \right) = O(T^2).
\]

As \( \delta \to 0 \). If, for example, \( \zeta(0) \to 0 \) for \( t > T_0 \), it follows that for \( T > T_0 \)
\[
\int_{T}^{T'} |Z(t)| \, dt = \int_{1}^{T'} \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt = O(T^2) \quad \text{as} \quad T \to \infty.
\]

This is inconsistent with (10.5.2), so that the theorem again fails.

10.6. Still another method depends on the formula (4.17.4), viz.
\[
Z(t) = 2 \sum_{n \leq N} \frac{\cos \left( \theta - \log n \right)}{\sqrt{n}} + O(t^2), \tag{10.6.1}
\]

where \( x = \sqrt{\log 2} \). Here \( \theta = \theta(t) \) is defined by
\[
x^{\frac{1}{2}} + it \to e^{\pi\nu \theta(t)},
\]

so that
\[
\theta(t) = -\frac{1}{2} \log \left( \frac{1}{2} + it \right) - \frac{1}{2} \log \left( 1 - \frac{1}{2} + it \right) - \frac{1}{2} \log \left( 1 + it \right) - \frac{1}{2} \log \left( 1 - it \right) = -\log \left( 1 + it \right) - \log \left( 1 - it \right) + \pi \log 2.
\]

The function \( \theta(t) \) is steadily increasing for \( t > T_0 \). If \( \nu \) is any positive integer \( (\geq \nu) \), the equation \( \theta(t) = \nu \pi \) therefore has just one solution, say \( t_\nu \), and \( t_\nu \sim 2\nu^2 \log \nu \).

Now
\[
Z(t_\nu) = 2 \sum_{n \leq \nu} \frac{\cos \left( \theta - \log n \right)}{\sqrt{n}} + O(t_\nu).
\]

The sum
\[
\sum_{n \leq \nu} \cos \left( \theta - \log n \right) = 1 + \cos(\pi \nu) + \cdots
\]

consists of the constant term unity and oscillatory terms; and the formula suggests that \( g(t) \) will usually be positive, and hence that \( Z(t) \) will usually change sign in the interval \( (t_\nu, t_{\nu+1}) \).

We shall prove
\[
\text{Theorem 10.6. As } N \to \infty \quad \sum_{n \leq N} Z(n) \sim 2N, \quad \sum_{n \leq N} Z(n+1) \sim -2N.
\]

It follows at once that \( Z(n) \) is positive for an infinity of values of \( n \), and that \( Z(n+1) \) is negative for an infinity of values of \( n \); and the existence of an infinity of real zeros of \( Z(t) \), and so of \( Z(t) \), again follows.

We have
\[
\sum_{n \leq N} \frac{\cos \left( \theta - \log n \right)}{\sqrt{n}} = \sum_{n \leq N} \frac{\cos \left( \theta - \log n \right)}{\sqrt{n}} - N - M - \sum_{1 \leq n \leq M} \frac{1}{n} \sum_{1 \leq k \leq \nu} \frac{\cos \left( \theta - \log n \right)}{\sqrt{n}}.
\]

where \( \nu = \max(kM+2, 2\nu^2) \). The inner sum is of the form
\[
\sum_{n \leq M} \cos(2\pi \nu n),
\]

where \( \nu = \max(kM+2, 2\nu^2) \). The inner sum is of the form
where \( \phi(x) = \frac{t_n \log n}{n} \).

We may define \( t_n \) for all \( x \geq x_4 \) (not necessarily integral) by \( \phi(t_n) = x \).

Then

\[
\phi'(x) = \frac{\log n}{x} \quad \text{and} \quad \phi''(t_n) = \frac{2 \log n}{t_n},
\]

so that

\[
\phi''(x) = \frac{2 \log n}{x^2}.
\]

Hence \( \phi''(x) \) is positive and steadily decreasing, and, if \( x \) is large enough,

\[
\phi''(x) = -2 \log n + 2 \log x \quad \text{with} \quad \phi''(t_n) \sim -2 \log x \quad \text{as} \quad x \to \infty.
\]

Hence, by Theorem 5.9,

\[
\sum_{x < \lambda \leq x + \Delta} \cos(t_n \log n) = O\left( \frac{\log x}{\log \log x} \right) + O\left( \frac{\log n}{\log x} \right).
\]

Hence

\[
\sum_{x < \lambda \leq x + \Delta} \frac{1}{x} \sum_{x < \lambda \leq x + \Delta} \cos(t_n \log n) = O\left( \frac{\log x}{\log \log x} \right) + O\left( \frac{\log n}{\log x} \right) = O(N \log N).
\]

Hence

\[
\int_{-T}^{T} \frac{L(t_n)}{N} \, dt = 2N + O(N \log N).
\]

and a similar argument applies to the other sum.

10.7. We denote by \( X_r(T) \) the number of zeros of \( \xi(s) \) of the form \( \frac{1}{2} + it \) \( (0 < t < T) \). The theorem already proved shows that \( X_r(T) \) tends to infinity with \( T \). We can, however, prove much more than this.

**Theorem 10.7** \( X_0(T) > AT \).

Any of the above proofs can be put in a more precise form so as to give results in this direction. The most successful method is similar in principle to that of § 10.6, but is more elaborate. We contrast the behaviour of the integrals

\[
I = \int_{-T}^{T} \frac{d^{-1} + \frac{1}{2} + \frac{1}{2} \pi \log \frac{T}{x}}{x} \, dt, \quad J = \int_{-T}^{T} \frac{\xi(\frac{1}{2} + \frac{1}{2} \pi \log d)}{x} \, dt,
\]

where \( T \leq t \leq 2T \) and \( T \to \infty \).

† Hardy and Littlewood (3).
Now
\[ \left| \sum \sum \frac{1}{x^{k/2} \log x} \right| \leq \frac{1}{2} \sum \sum \frac{1}{x^{k/2} \log x} \]
As in § 10.5, the first sum is \( O(x^{-1/2} \log x) \), and its contribution to (10.7.3) is therefore
\[ O \left( \int_{\frac{1}{2}}^{1} x^{-1/2} \, dx \right) + O \left( \int_{\frac{1}{2}}^{1} x^{-1/2} \, dx \right) \log \log x \]
\[ = O(H^{-1/2}) + O(\log \log x) = O(H^{-1}). \]
The sum with \( m \neq s \) contributes to the second term in (10.7.3) terms of the form
\[ \int_{\frac{1}{2}}^{1} \frac{1}{x^{k/2} \log x} \, dx = O \left( \frac{1}{x^{k/2} \log x} \right) \]
by Lemma 4.3. Hence the sum is
\[ O \left( \int_{\frac{1}{2}}^{1} \frac{1}{x^{k/2} \log x} \, dx \right) + O \left( \int_{\frac{1}{2}}^{1} \frac{1}{x^{k/2} \log x} \, dx \right) \log \log x \]
\[ = O \left( \int_{\frac{1}{2}}^{1} \frac{1}{x^{k/2} \log x} \, dx \right) \log x \]
for \( \delta < \delta(H) \). The first integral in (10.7.3) may be dealt with in the same way. Hence
\[ \int_{\frac{1}{2}}^{1} \frac{1}{x^{k/2} \log x} \, dx = O(H^{-1}). \]
Taking \( \delta = 1/2 \) and \( T > T_0(H) \), it follows that
\[ \int_{\frac{1}{2}}^{1} x^{1/2} \, dt = O(HT^4). \]
(10.7.3)

10.8. We next prove that
\[ J > (AH + W)T^{-1}, \]
where
\[ \int_{\frac{1}{2}}^{1} x^{1/2} \, dt = O(1) \quad (0 < H < 2). \]
(10.8.1)

We have, if \( a = \frac{1}{2} + it, \ T < t < 2T, \)
\[ T^{-2} \| \theta(t) \|^2 < A(k_{1} + 1). \]
Hence
\[ T \delta > A \int_{\frac{1}{2}}^{1} \frac{\| \theta(t) \|^2}{(\delta + 1) \log \| \theta(t) \|^2} \, dt \]
\[ = A \left( \int_{\frac{1}{2}}^{1} \frac{1}{|t - 1/k_{1}|} + O(T^{-1}) \right) \, dt \]
\[ = AH + O \left( \int_{\frac{1}{2}}^{1} \sum_{c \neq s \in H} \left| \frac{1}{\delta + 1} \right| \, dt \right) \log \log x \]
\[ = AH + O \left( \sum_{c \neq s \in H} \frac{1}{|c - 1/k_{1}|} \log \log n \right) = O(H^4). \]

It is now sufficient to prove that
\[ \int_{\frac{1}{2}}^{1} \frac{1}{|t - 1/k_{1}|} \, dt = O(T), \]
and the calculations are similar to those of § 7.3, but with an extra factor \( \log \log n \) in the denominator.

To prove Theorem 10.7, let \( S \) be the sub-set of the interval \((T, 2T)\) where \( I = J \). Then
\[ \int_{\frac{1}{2}}^{1} |I| \, dt = \int_{J} |I| \, dt. \]
Now
\[ \int_{\frac{1}{2}}^{1} |I| \, dt \ll \int_{\frac{1}{2}}^{1} |I| \, dt \ll \int_{\frac{1}{2}}^{1} |I| \, dt \ll AHT^4 \]
by (10.7.3); and by (10.8.1) and (10.8.2)
\[ \int_{\frac{1}{2}}^{1} \left| T \right| \, dt > T^{-2} \int_{\frac{1}{2}}^{1} (AH + W) \, dt \]
\[ > AT^{-2} \left( Hm(S) - T^{-1} \right) \left| \Psi \right| \, dt \]
\[ > AT^{-2} \left( Hm(S) - T^{-1} \left| \Psi \right| \right) \]
\[ > AT^{-2} \left( Hm(S) - T^{-1} \right) \]
where \( m(S) \) is the measure of \( S \). Hence, for \( H > 1 \) and \( T > T_0(H) \),
\[ m(S) < AHT^4. \]
Now divide the interval \((T, 2T)\) into \([T, 2H]\) pairs of abutting intervals \(j_1, j_2\), each, except the last \(j_n\), of length \(H\), and each \(j_n\) lying to the right of the corresponding \(j_n\). Then either \(j_1\) or \(j_2\) contains a zero of \(\zeta(s)\) unless \(j_n\) consists entirely of points of \(S\). Suppose that the latter occurs for \(\mathcal{N}(s)\). Then
\[
\nu H \leq m(\mathcal{N}) < A TH^{-\frac{1}{2}}.
\]
Hence there are, in \((T, 2T)\), at least
\[
[T/2H] - \nu > \frac{T}{H} \left( 1 - \frac{A}{\sqrt{H}} \right) > \frac{T}{4H}
\]
zeros if \(H\) is large enough. This proves the theorem.

10.9. For many years the above theorem of Hardy and Littlewood, that \(N(T) > AT\), was the best that was known in this direction. Recently it has been proved by A. Selberg that \(N(T) > AT \log T\). This is a remarkable improvement, since it shows that a finite proportion of the zeros of \(\zeta(s)\) lie on the critical line. On the Riemann hypothesis, of course,
\[
N(T) = N(T) - \frac{1}{2\pi} T \log T.
\]
The numerical value of the constant \(A\) in Selberg's theorem is very small.

The essential idea of Selberg's proof is to modify the series for \(\zeta(s)\) by multiplying it by the square of a partial sum of the series for \((\zeta(s))^{-1}\). To this extent, it is similar to the proof given in Chapter 7 of Weinstock's book on the general distribution of the zeros.

We define \(\alpha_n\) by
\[
\frac{1}{\sqrt{\gamma(s)}} = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s} \quad (\sigma > 1), \quad \alpha_1 = 1.
\]
It is seen from the Euler product that \(\sum_{n=1}^{\infty} \alpha_n^{-1} = \zeta(\sigma)\) if \((\sigma, r) = 1\). Since the series for \((1 - s)^{-1}\) is majorized by that for \((1 - s)^{-1}\), we see that, if
\[
\sqrt{\gamma(s)} = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^s},
\]
then \(|\alpha_n| |\alpha_n| < 1\).

Let
\[
\beta_n = \alpha_n \left( 1 - \frac{\log n}{\log \frac{n}{X}} \right) \quad (1 < n < X).
\]
Then
\[
|\beta_n| < 1.
\]

10.10. Let
\[
\Phi(s) = \frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\psi(v)}{v} \zeta(s) \zeta(1-s) ds
\]
where \(c > 1\). Moving the line of integration to \(s = \frac{1}{2}\), and evaluating the residue at \(s = 1\), we obtain
\[
\Phi(s) = -\frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\psi(v)}{v} \zeta(s) \zeta(1-s) ds
\]
\[
= -\frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(1-s) \psi(v)}{v} ds.
\]

On the other hand,
\[
\Phi(s) = \frac{1}{4\pi} \sum_{\nu \neq 0} \sum_{\nu \neq 0} \frac{\beta_\nu}{\nu} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\psi(v)}{v} \exp \left( -\frac{\pi \nu^2 y^2}{\nu y^2} \right) v ds.
\]

Putting \(s = e^{-2\pi y^2}, \nu\), it follows that the functions
\[
F(t) = \frac{1}{\sqrt{2\pi i}} \left( 1 + i e^{-\frac{\pi y^2}{4}} \right)\Phi(s) dt
\]
are Fourier transforms. Hence, as seen in §10.7,
\[
\int_{\frac{1}{2}}^{\frac{1}{2} + a} \left( \frac{1}{4\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \frac{\psi(v)}{v} \zeta(s) \zeta(1-s) ds \right) dt \leq C T^{a/2} \int_{\frac{1}{2}}^{\frac{1}{2} + a} |f(y)| y^{-a} dy + 8 \int_{\frac{1}{2}}^{\infty} |f(y)| y^{-a} dy
\]
where \(a < 1\) is to be chosen later.

Putting \(y = \log x, G = e^{-\frac{\pi y^2}{4}}\), the first integral on the right is equal to
\[
\int_{\frac{1}{2}}^{\frac{1}{2} + a} e^{-2\pi y^2} \psi(v) \zeta(s) \zeta(1-s) ds = \sum_{\nu \neq 0} \sum_{\nu \neq 0} \frac{\beta_\nu}{\nu} \exp \left( -\frac{\pi \nu^2 y^2}{4} \right) v ds,
\]
\(\dagger\) Titchmarsh (26).
Calling the triple sum $g(p)$, this is not greater than
\[
\frac{1}{2} \int \frac{|\phi(1)\phi(0)|^3}{|x-z|^2} \, dx + 2 \int \frac{|\phi(z)|^3}{|x-z|^2} \, dx < \frac{1}{2} \int |\phi(1)\phi(0)|^3 \, dx + 2 \int |\phi(z)|^3 \, dx.
\]
Similarly, the second integral in (10.10.1) does not exceed
\[
\frac{|\phi(1)\phi(0)|^3}{20\log^2|z|} + \frac{1}{2} \int |\phi(z)|^3 \, dx.
\]

10.11. We have to obtain upper bounds for these integrals as $\delta \to 0$, but it is more convenient to consider directly the integral
\[
J(x, \theta) = \int y^\nu |\nu|^{-d} \, dy \quad (0 < \theta \ll \frac{1}{2}, \quad x \gg 1).
\]
This is equal to
\[
\sum_{\nu} \sum_{\nu} \sum_{\nu} \beta_{\nu} \beta_{\nu} \beta_{\nu} \frac{1}{\nu \theta} \int e^{\nu(i\theta \sin 5 + \nu \cos 5)} \nu^{\nu |\nu|^{-d}} \, d\nu.
\]
Let $\Sigma_{\nu}$ denote the sum of those terms in which $\nu_{\nu} = \nu_{\nu} = \nu_{\nu}$, and $\Sigma$ the remainder. Let $(\nu, \lambda, \mu) = (\nu, \lambda, \mu)$ so that $\nu = \nu_{\nu}$, $\lambda = \lambda_{\lambda}$, $\mu = \mu_{\mu}$.

Then, in $\Sigma_{\nu}$, $m_{\nu} = n_{\nu}$, so that $n_{\nu} = n_{\nu}$, $m_{\nu} = m_{\nu}$ (r = 1, 2, ...). Hence
\[
\Sigma_{\nu} = \sum_{\nu} \beta_{\nu} \beta_{\nu} \beta_{\nu} \frac{1}{\nu \theta} \int e^{\nu(i\theta \sin 5 + \nu \cos 5)} \nu^{\nu |\nu|^{-d}} \, d\nu.
\]
Now
\[
\sum_{\nu} \frac{\beta_{\nu} \beta_{\nu} \beta_{\nu}}{\nu \theta} \int e^{\nu(i\theta \sin 5 + \nu \cos 5)} \nu^{\nu |\nu|^{-d}} \, d\nu = \frac{1}{\theta} \int \frac{e^{|\nu|^{-d} \nu \cos 5}}{\nu^{\nu |\nu|^{-d}}} \, d\nu.
\]
The last $r$-sum is of the form
\[
\frac{1}{\theta} \int \frac{e^{|\nu|^{-d} \nu \cos 5}}{\nu^{\nu |\nu|^{-d}}} \, d\nu + K(\theta) + O\left(\frac{1}{\theta}ight),
\]
where $K(\theta)$, and later $K(\theta)$, are bounded functions of $\theta$. Hence we obtain
\[
\begin{align*}
\frac{1}{\theta} & \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu + \sum_{\nu \nu} \beta_{\nu} \beta_{\nu} \beta_{\nu} \frac{1}{\nu \theta} \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu \\
= \frac{1}{\theta} & \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu + \sum_{\nu \nu} \beta_{\nu} \beta_{\nu} \beta_{\nu} \frac{1}{\nu \theta} \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu.
\end{align*}
\]
Putting $\eta = 2\nu_{\nu} \nu_{\nu} \nu_{\nu} \sin 5$, it follows that
\[
\Sigma_{\nu} = \frac{1}{\theta} \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu + \sum_{\nu \nu} \beta_{\nu} \beta_{\nu} \beta_{\nu} \frac{1}{\nu \theta} \int e^{|\nu|^{-d} \nu \cos 5} \, d\nu.
\]
10.11. Lemma 10.12. We have
\[
\sum_{\nu \nu} \frac{\nu_{\nu} \nu_{\nu} \nu_{\nu}}{\nu \theta} \log \frac{X}{\delta} = O\left(\frac{1}{\theta}ight),
\]
ununiformly with respect to $\theta$.  

\[
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\]
We may suppose that $X \geq 2M$, since otherwise the lemma is trivial.

Now
\[ \frac{1}{2\pi i} \int_{\gamma} \frac{\pi}{x^2} \, ds = 0 \quad \left(0 < x < 1, \quad \log \frac{x}{x+1} \right). \]

Also
\[ \sum_{\nu \in \mathbb{C} \setminus \mathbb{R}} \frac{r \gamma_{\nu}}{x^2 - \pi^2 \nu^2} = \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 - \frac{1}{x^2 - \pi^2 \nu^2}\right) = \sum_{\nu \in \mathbb{R}} \left(1 - \frac{1}{x^2 - \pi^2 \nu^2}\right)^{-1} \frac{1}{\sqrt{\nu(1 - \nu + \delta)}}. \]

Hence the left-hand side of (10.12) is equal to
\[ \frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{x^2 - \pi^2 \nu^2}\right)^{-1} \frac{ds}{\sqrt{\nu(1 - \nu + \delta)}}. \] (10.12.2)

There are singularities at $s = 0$ and $s = \delta$. If $\delta > (\log(|X|))/4$, we can take the line of integration through $s = \delta$, the integral round a small indentation tending to zero. Now
\[ \begin{aligned}
\left| \frac{1}{X(1+\delta)} \right| & < A(t) \\
\left| \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right| & < O(1) \\
\end{aligned} \]

Hence (10.12.2) is
\[ O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \int_{\gamma} \frac{ds}{\delta + \nu^2}\right) = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

and the result stated follows.

If $\delta < (\log(|X|))/4$, we take the same contour as before modified by a detour round the right-hand side of the circle $|s| = 2(\log(|X|))/4$. On this circle
\[ |X(1+\delta)| < \epsilon, \]
the $p$-product goes as before, and
\[ |X(1-\delta + \epsilon)| > 4 \log(|X|)/\delta. \]

Hence the integral round the circle is
\[ O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \mathcal{P}\right) = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

The integral along the part of the line $\mathcal{P} = \mathcal{P}$ above the circle is
\[ O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

The lemma is thus proved in all cases.

10.13. **Lemma 10.13.**
\[ \sum_{\nu \in \mathbb{R} \setminus \mathbb{R}} \frac{|a_X|}{\delta + \nu^2} = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

Defining $\mathcal{A}_\mathcal{P}$ as in §10.9, we have
\[ \sum_{\nu \in \mathbb{R} \setminus \mathbb{R}} \frac{|a_X|}{\delta + \nu^2} \leq \sum_{\nu \in \mathbb{R} \setminus \mathbb{R}} \frac{|a_X|}{\delta + \nu^2} = \sum_{\nu \in \mathbb{R} \setminus \mathbb{R}} \frac{1}{\nu^2}. \]

Where $D$ is a number of the same class as $d$ or $d_1$,
\[ \frac{1}{\nu} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

10.14. **Lemma 10.14.**
\[ S(\theta) = O\left(\frac{X^4}{\delta} \right) \]

uniformly with respect to $\theta$. In particular,
\[ S(\theta) = \frac{\theta}{\log X} \]

By the formula of §10.11, and the above lemmas,
\[ \sum_{\nu \in \mathbb{R} \setminus \mathbb{R}} \frac{\theta}{\nu^2} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]

Hence
\[ S(\theta) = O\left(\frac{X^4}{\delta} \prod_{\nu \in \mathbb{R} \setminus \mathbb{R}} \left(1 + \frac{1}{x^2 - \pi^2 \nu^2}\right) \right). \]
Hence
\[
S(\beta) = O\left( \frac{X^\delta}{\log^2 X} \sum_{n \leq X} \sum_{\substack{\mu \leq n \leq X \delta}} \frac{1}{m \phi(n) \mu^{1+ \varepsilon}} \right)
\]
\[
= O\left( \frac{X^\delta}{\log^2 X} \sum_{n \leq X} \sum_{\mu \leq n} \frac{1}{m \phi(n) \mu^{1+ \varepsilon}} \right)
\]
\[
= O\left( \frac{X^\delta}{\log^2 X} \sum_{n \leq X} \frac{1}{m \phi(n)} \right)
\]
\[
= O\left( \frac{X^\delta}{\log X} \right).
\]

10.15. Estimation of \( \Sigma \). From (10.11.1), Lemma 10.14, and the inequality \( |\beta| \leq 1 \), we obtain
\[
\Sigma_n = O\left( \frac{1}{\delta \log^2 X} \right) + O\left( \frac{\delta \log X}{\delta \log^2 X} \right) + O\left( \frac{X^\delta}{\log^2 X} \right).
\]

We shall ultimately take \( X = \delta^{-1} \) and \( a = (a \log X)^{\delta} \), where \( a \) and \( \delta \) are suitable positive constants. Then \( X = X^{\delta^{-1}} \). If \( x \geq G \), the last two terms can be omitted in comparison with the first if \( G X = O(\delta^{-1}) \).

\[
\Sigma_n = O\left( \frac{1}{\delta \log^2 X} \right) + O\left( \frac{1}{\delta \log^2 X} \right) + O\left( \frac{X^\delta}{\log^2 X} \right).
\]

10.16. Estimation of \( \Sigma_\theta \). If \( P \) and \( Q \) are positive, and \( x \geq 1 \),
\[
\int_0^x e^{-(s \log x)^{\delta}} ds = \frac{1}{\delta} \int_0^{x^{\delta}} e^{-u^{\delta}} dt - O\left( \frac{x^{\delta}}{\delta \log^2 X} \right),
\]

for example,
\[
\int_0^\infty e^{-x^2} \cos(x \log x) dx = \frac{1}{2} \int_0^{\infty} e^{-u^2} du - O\left( \frac{1}{\delta \log^2 X} \right).
\]

The terms with \( m \mu \beta \geq \mu \delta \) contribute to the \( m \), \( n \) sum
\[
O\left( \sum_{m \leq x} \sum_{\mu \leq n} \frac{m \mu \beta^{1+ \varepsilon}}{\mu \phi(n) \mu^{1+ \varepsilon}} \right)
\]
\[
= \frac{m \mu \beta^{1+ \varepsilon}}{\mu \phi(n) \mu^{1+ \varepsilon}} - \frac{m \mu \beta^{1+ \varepsilon}}{\mu \phi(n) \mu^{1+ \varepsilon}} \frac{m \mu \beta^{1+ \varepsilon}}{\mu \phi(n) \mu^{1+ \varepsilon}}
\]
and
\[
\sum_{m \mu \beta \geq m \mu \delta} \frac{1}{m \phi(n) \mu^{1+ \varepsilon}} \geq 1 + \frac{1}{m \phi(n) \mu^{1+ \varepsilon}} + \cdots + 1 = O\left( \frac{1}{m \phi(n) \mu^{1+ \varepsilon}} \right).
\]

10.17. Lemma 10.17. Under the assumptions of § 10.15
\[
\int_a^b P(x) dx = O\left( \frac{1}{\delta \log^2 X} \right).
\]

By (10.15.1) and (10.16.1),
\[
J(x, \theta) = O\left( \frac{1}{\delta \log^2 X} \right)\]

uniformly with respect to \( \theta \). Hence
\[
\frac{1}{\theta} \log \left( \frac{1}{\delta \log X} \right) + O\left( \frac{1}{\delta \log^2 X} \right) = O\left( \frac{1}{\delta \log^2 X} \right).
\]

taking, for example, \( \theta = \frac{1}{2} \). Also
\[
\int_0^\infty \log \left( \frac{1}{\delta \log^2 X} \right) dx = \frac{1}{\delta} \int_0^\infty \log x dx
\]
\[
= \frac{1}{\delta} \int_0^\infty \frac{x \log x}{x^2 \log^2 x} dx
\]
\[
\geq \frac{1}{2} \frac{1}{\delta} \left( \int_0^\infty \log^2 x dx - \frac{3}{2} \int_0^\infty \frac{x \log x}{x^2} dx \right)
\]
since $G = e^{|b|}$. Hence

\[ \int \frac{|g(z)|^2}{\log^2 |z|} \, dz \leq \frac{1}{\delta} \psi / (\log X) \delta^2 + \frac{1}{\delta} \psi / (\log X) \delta^2 = O\left( \frac{1}{\delta^2 \log^2 |\log X|} \right). \]

Also $\phi(0) = O(X)$, $\phi(1) = O(\log X)$. The result therefore follows from the formula of § 10.10.

10.18. So far the integrals considered have involved $F(t)$. We now turn to the integrals involving $|F(t)|$. The results about each integral are expressed in the following lemma.

**Lemma 10.18.** \[ \int \left| \int \frac{1}{\log X} \right| \, dt = O\left( \frac{1}{\log X} \right). \]

By the Fourier transform formula, the left hand side is equal to

\[ \int \left( \frac{1}{\log X} \right) \, dt = \int \int e^{-2\pi i x y} \left( \frac{1}{\log X} \right) \, dx \, dy \]

Taking $x = 1$, $\theta = (\log(1/\delta))^{-1}$ in (10.17.2), we have

\[ \int \left| \int \frac{1}{\log X} \right| \, dt = O\left( \frac{1}{\log X} \right). \]

Hence

\[ \int \left| \int \frac{1}{\log X} \right| \, dt = O\left( \frac{1}{\log X} \right). \]

We can estimate the integral over $(3,\infty)$ in a comparatively trivial manner. As in § 10.11, this is less than

\[ \sum_{n=1}^\infty \sum_{\delta=1}^\infty \sum_{\delta=1}^\infty \frac{\delta \beta_0 \beta_1 \delta}{\delta^2 \beta} \exp \left\{ -2 \left( \frac{\delta^2 \beta_0^2}{\delta^2 \beta} + \frac{\delta^2 \beta_1^2}{\delta^2 \beta} \right) \sin \delta \right\} \, d\delta. \]

Using, for example, $e^{\delta^2 \beta} \sin \delta > AX^{-\delta} > AB^2$ (since $x = \delta^{-\epsilon}$ with $\epsilon = \frac{1}{2}$), and $|\beta| < 1$, this is

\[ O\left( X^2 \log^2 X \sum_{\delta=1}^\infty \sum_{\delta=1}^\infty \int e^{-2\delta^2 \beta_0^2 \sin \delta} \, d\delta \right) = O\left( X^2 \log^2 X \right) \]

which is of the required form.

10.19. **Lemma 10.19.**

\[ \int \left( \frac{1}{\log X} \right) \, dt = O\left( \frac{1}{\log X} \right). \]

For the left-hand side does not exceed

\[ \int \left( \frac{1}{\log X} \right) \, dt = \int \left( \frac{1}{\log X} \right) \, dt = \int \left( \frac{1}{\log X} \right) \, dt, \]

and the result follows from the previous lemma.

10.20. **Lemma 10.20.**

If $\beta = 1/T$,

\[ \int \left( \frac{1}{\log X} \right) \, dt = O\left( \frac{1}{\log X} \right). \]

We have

\[ \sum_{\delta=1}^\infty \sum_{\delta=1}^\infty \sum_{\delta=1}^\infty \int e^{-2\delta^2 \beta_0^2 \sin \delta} \, d\delta = \frac{1}{\beta} \int \frac{1}{\log X} \, dt = \frac{1}{\beta} \int \frac{1}{\log X} \, dt, \]

Since $\phi(\epsilon) = O(X)$ for $\epsilon > \frac{1}{2}$, the first term is $O(X)$, and the third is $O(1)$. Also

\[ \zeta(\epsilon) \phi(\epsilon) = 1 + \sum_{n=1}^\infty \frac{a_n}{\epsilon^n}, \]

where $|a_n| < 2\epsilon(n)$. Hence

\[ \int \frac{1}{\log X} \, dt = \int \frac{1}{\log X} \, dt = \int \frac{1}{\log X} \, dt, \]

It follows that

\[ \int \left( \frac{1}{\log X} \right) \, dt \sim T. \]

\[ \int_0^{\lambda} \int_0^{b} |F(u)| \, du > \lambda \int_0^{b} |F(u)| \, du > \lambda T. \]

The left-hand side is equal to

\[ \int_0^{\lambda} \int_0^{b} |F(u)| \, du \int_0^{\lambda} \int_0^{b} \frac{d}{dt} |F(u)| \, du \, dt = \lambda \int_0^{b} |F(u)| \, du, \]

and the result follows from the previous lemma.

10.22. Theorem 10.22.

\[ \Lambda(T) > A \lambda T \log T. \]

Let \( E \) be the subset of \((0, T)\) where

\[ \int_0^{\lambda} |F(u)| \, du > \lambda \int_0^{b} |F(u)| \, du. \]

For such values of \( t \), \( F(u) \) must change sign in \((t, t + h)\), and hence so must \( \Xi(u) \), and hence \( (1 + iu) \) must have a zero in this interval.

Since the two sides are equal except in \( E \),

\[ \int_0^{\lambda} \int_0^{b} |F(u)| \, du \int_0^{\lambda} \int_0^{b} \frac{d}{dt} |F(u)| \, du \, dt = \lambda \int_0^{b} |F(u)| \, du, \]

\[ > \lambda \int_0^{b} |F(u)| \, du \int_0^{\lambda} |F(u)| \, du \, dt. \]

The left-hand side is not greater than

\[ \left( \lambda \int_0^{b} |F(u)| \, du \right)^2 < \left( (m(E))^2 \int_0^{b} |F(u)| \, du \right)^2 \leq A \lambda T \int_0^{\lambda} |F(u)| \, du \]

by Lemma 10.19 with \( \delta = 1/T \). The second term on the right is not greater than

\[ \left( \lambda \int_0^{b} |F(u)| \, du \right)^2 < \lambda A \lambda T \int_0^{\lambda} \frac{d}{dt} |F(u)| \, du \]

by Lemma 10.17. Hence

\[ \left( m(E) \right)^2 > A \lambda T \int_0^{\lambda} \frac{d}{dt} |F(u)| \, du \]

where \( A_4 \) and \( A_4 \) denote the particular constants which occur. Since \( X = T \) and \( h = (\log T)^{-1} \),

\[ \left( m(E) \right)^2 > A \lambda e T \int_0^{\lambda} \frac{d}{dt} |F(u)| \, du \]

Taking a small enough, it follows that

\[ m(E) > A \lambda T. \]

Hence, of the intervals \((0, \lambda), (\lambda, 2\lambda), \ldots \) contained in \((0, T)\), at least \( [A_4 T] \) must contain points of \( E \). If \((x, (x + 1)) \) contains a point \( t \) of \( E \), there must be a zero of \( (1 + iu) \) in \((t, t + h)\), and so in \((y, (y + 2)) \).

Allowing for the fact that each zero might be counted twice in this way, there must be at least

\[ \lambda \int_0^{\lambda} |F(u)| \, du > A \lambda T \log T \]

zeros in \((0, T)\).

10.23. In this section we return to the function \( \Xi^*(t) \) mentioned in § 10.1. In spite of its deficiencies as an approximation to \( \Xi(t) \), it is of some interest to note that all the zeros of \( \Xi^*(t) \) are real.

A still better approximation to \( \Xi(u) \) is

\[ \Xi^**(u) = \sum \Im \frac{1}{F(u) \cos \frac{\pi x}{T}} \cos \frac{\pi x}{T}, \]

and we shall also prove that all the zeros of \( \Xi^**(t) \) are real.

The function \( \Xi^*(u) \) is, for any value of \( u \), an even integral function of \( u \). We begin by proving that \( u \) is a real all its zeros are purely imaginary.

It is known that \( w = \Xi^*(u) \) satisfies the differential equation

\[ \frac{d^2}{du^2} w - \frac{d}{du} w + (a + \frac{\pi^2}{T}) w. \]

This is equivalent to the two equations

\[ \frac{d}{du} w - \frac{W}{u} = (a + \frac{\pi^2}{T}) w, \]

\[ \frac{W}{u} = \Xi^*(u) w. \]

† Fairy (1), (3), (4).
These give
\[ \frac{\partial}{\partial a}(W^0) = \frac{1}{a} \left( |W^0(a^2+x^2)|^2 |w|^4 \right). \]

It is also easily verified that \( W \) and \( W^0 \) tend to 0 as \( a \to \infty \). It follows that, if \( w \) vanishes for a certain 2 and \( a = a_0 > 0 \), then
\[ \int_a^{\infty} \left( |W|^2 + |a^2 + x^2|^2 |w|^4 \right) \frac{da}{a} = 0. \]

Taking imaginary parts,
\[ 2ix \int \frac{\partial |W|^2}{\partial a} \frac{da}{a} = 0. \]

Here the integral is not 0, and \( K_0(x) \) plainly does not vanish for \( a \) real, i.e. \( y = 0 \). Hence \( z = 0 \), the required result.

We also require the following lemma.

Let \( c \) be a positive constant, \( F(z) \) an integral function of genus 0 or 1, which takes real values for real \( z \), and has no complex zeros and at least one real zero. Then all the zeros of
\[ F(z+ic) + F(z-ic) \]
are also real.

We have
\[ F(z) = C e^{\alpha z} \sum_{n=1}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
where \( C, \alpha, \alpha_n \ldots \) are real constants, \( \alpha_n \neq 0 \) for \( n = 1, 2, \ldots \), \( \sum \alpha_n^{-1} \) is convergent, \( y \) a non-negative integer. Let \( z \) be a zero of (10.231). Then
\[ |F(z+ic)| = |F(z-ic)|, \]
so that
\[ 1 = \frac{|F(z+ic)|^2}{|F(z+ic)|^2} = \frac{(x^2 + (y+c)^2)^2}{(x^2 + (y-c)^2)^2} \sum_{n=1}^{\infty} \left( \frac{y+c}{\alpha_n} \right)^{\alpha_n} \]
and, if \( y > 0 \), every factor on the right is < 1; if \( y < 0 \), every factor is > 1. Hence in fact \( y = 0 \).

The theorem that the zeros of \( \Xi^*(c) \) are all real now follows on taking
\[ F(z) = 1/\alpha_n(2n), \quad c = 2 \]
10.24. For the discussion of \( \Xi^*(c) \) we require the following lemma.

Let \( f(t) < K e^{-\lambda t} \) for some positive \( \lambda \), so that
\[ F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{zt} dt \]
is an integral function of \( z \). Let all the zeros of \( F(z) \) be real. Let \( \psi(t) \) be an integral function of \( z \) of genus 0 or 1, real for real \( t \). Then the zeros of
\[ G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\psi(t)e^{zt} dt \]
are also all real.

We have
\[ \psi(t) = C e^{\alpha t} \sum_{n=1}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
where the constants are all real, and \( \sum \alpha_n^{-1} \) is convergent. Let
\[ \delta_n(t) = \int_{-\infty}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
here the integral is not 0, and \( K_0(x) \) plainly does not vanish for \( x \) real, i.e. \( y = 0 \). Hence \( z = 0 \), the required result.

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Let \( c \) be a positive constant, \( F(z) \) an integral function of genus 0 or 1, which takes real values for real \( z \), and has no complex zeros and at least one real zero. Then all the zeros of
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\[ |F(z+ic)| = |F(z-ic)|, \]
so that
\[ 1 = \frac{|F(z+ic)|^2}{|F(z+ic)|^2} = \frac{x^2 + (y+c)^2}{x^2 + (y-c)^2} \sum_{n=1}^{\infty} \left( \frac{y+c}{\alpha_n} \right)^{\alpha_n} \]
and, if \( y > 0 \), every factor on the right is < 1; if \( y < 0 \), every factor is > 1. Hence in fact \( y = 0 \).

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\[ G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\psi(t)e^{zt} dt \]
are also all real.

We have
\[ \psi(t) = C e^{\alpha t} \sum_{n=1}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
where the constants are all real, and \( \sum \alpha_n^{-1} \) is convergent. Let
\[ \delta_n(t) = \int_{-\infty}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
here the integral is not 0, and \( K_0(x) \) plainly does not vanish for \( x \) real, i.e. \( y = 0 \). Hence \( z = 0 \), the required result.

We also require the following lemma.

Let \( c \) be a positive constant, \( F(z) \) an integral function of genus 0 or 1, which takes real values for real \( z \), and has no complex zeros and at least one real zero. Then all the zeros of
\[ F(z+ic) + F(z-ic) \]
are also real.

We have
\[ F(z) = C e^{\alpha z} \sum_{n=1}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
where \( C, \alpha, \alpha_n \ldots \) are real constants, \( \alpha_n \neq 0 \) for \( n = 1, 2, \ldots \), \( \sum \alpha_n^{-1} \) is convergent, \( y \) a non-negative integer. Let \( z \) be a zero of (10.231). Then
\[ |F(z+ic)| = |F(z-ic)|, \]
so that
\[ 1 = \frac{|F(z+ic)|^2}{|F(z+ic)|^2} = \frac{x^2 + (y+c)^2}{x^2 + (y-c)^2} \sum_{n=1}^{\infty} \left( \frac{y+c}{\alpha_n} \right)^{\alpha_n} \]
and, if \( y > 0 \), every factor on the right is < 1; if \( y < 0 \), every factor is > 1. Hence in fact \( y = 0 \).

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\[ G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\psi(t)e^{zt} dt \]
are also all real.

We have
\[ \psi(t) = C e^{\alpha t} \sum_{n=1}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
where the constants are all real, and \( \sum \alpha_n^{-1} \) is convergent. Let
\[ \delta_n(t) = \int_{-\infty}^{\infty} \left( 1 - \frac{\varepsilon}{\alpha_n} \right)^{\alpha_n} \]
here the integral is not 0, and \( K_0(x) \) plainly does not vanish for \( x \) real, i.e. \( y = 0 \). Hence \( z = 0 \), the required result.

We also require the following lemma.

Let \( c \) be a positive constant, \( F(z) \) an integral function of genus 0 or 1, which takes real values for real \( z \), and has no complex zeros and at least one real zero. Then all the zeros of
\[ F(z+ic) + F(z-ic) \]
are also real.
then \( G(c) = \mathbb{Z}^*(c) \), and it follows again that all the zeros of \( \Sigma^*(s) \) are real. If
\[
\phi(s) = \frac{\pi}{2} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)
\]
then \( G(c) = \mathbb{Z}^*(c) \). Hence all the zeros of \( \Sigma^*(c) \) are real.

10.25. By way of contrast to the Riemann zeta-function we shall now construct a function which has a similar functional equation, and for which the analogues of most of the theorems of this chapter are true, but which has no Euler product, and for which the analogue of the Riemann hypothesis is false.

We shall use the simplest properties of Dirichlet’s \( L \)-functions (mod 5). These are defined for \( s > 1 \) by
\[
L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{1}{1 - \frac{1}{2^s}} \frac{1}{1 + \frac{1}{2^s}} \frac{1}{1 + \frac{1}{3^s}} \frac{1}{1 + \frac{1}{4^s}} \cdots,
\]
\[
L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{1}{1 - \frac{1}{2^s}} \frac{1}{1 + \frac{1}{2^s}} \frac{1}{1 + \frac{1}{3^s}} \frac{1}{1 + \frac{1}{4^s}} \cdots,
\]
\[
L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{1}{1 - \frac{1}{2^s}} \frac{1}{1 + \frac{1}{2^s}} \frac{1}{1 + \frac{1}{3^s}} \frac{1}{1 + \frac{1}{4^s}} \cdots,
\]
Each \( \chi(n) \) has the period 5. It is easily verified that in each case
\[
\chi(n) \chi(n) = \chi(n+5)
\]
if \( n \) is prime to 5; and hence that
\[
L(s) = \prod_{\text{prime } p \neq 5} \left( 1 - \frac{1}{p^s} \right)^{-1} \quad (s > 1).
\]

It is also easily seen that
\[
L(s) = \prod_{\text{prime } p \neq 5} \left( 1 - \frac{1}{p^s} \right)^{-1}
\]
so that \( L(s) \) is regular except for a simple pole at \( s = 1 \). The other three series are convergent for any real positive \( s \), and hence for \( s > 0 \).

Hence \( L(s) \), \( L(s) \), and \( L(s) \) are regular for \( s > 0 \).

Now consider the function
\[
f(s) = \frac{1}{s} \sin \left( \frac{\pi}{5} s \right) \left( 1 - \frac{1}{5^s} \right) L(s) + \sin \left( \frac{\pi}{5} s \right) L(s),
\]
\[
= \frac{1}{s} \left( 1 - \frac{1}{5^s} \right) \left( 1 + \frac{1}{2^s} \right) \left( 1 - \frac{1}{3^s} \right) \left( 1 + \frac{1}{4^s} \right) \left( 1 - \frac{1}{5^s} \right) \cdots
\]
\[
= \frac{1}{s} \left( 1 + \tan \left( \frac{\pi}{5} s \right) \left( 1 - \frac{1}{5^s} \right) \left( 1 + \frac{1}{2^s} \right) \left( 1 - \frac{1}{3^s} \right) \left( 1 + \frac{1}{4^s} \right) \left( 1 - \frac{1}{5^s} \right) \right.
\]
where \( \zeta(e, s) \) is defined as in \( \S 2.17 \).
If \( p \) is a prime, we define \( a(p) \) by
\[
a(p) = \frac{1}{2}(1+i)\chi_1(p) + \frac{1}{2}(1-i)\chi_2(p),
\]
so that
\[
a(p) = \pm 1 \quad \text{or} \quad \pm i.
\]
For composite \( n \), we define \( a(n) \) by the equation
\[
a(n) = a(n_1)n_2(a_1).
\]
Thus \( |a(n)| \) is always 0 or 1. Let
\[
M(s, \chi) = \sum_{n=1}^\infty \frac{a(n)\chi(n)}{n^s} = \prod_p \left(1 - \frac{a(p)\chi(p)}{p^s}\right)^{-1},
\]
where \( \chi \) denotes either \( \chi_1 \) or \( \chi_2 \). Let
\[
N(s) = \frac{1}{2}(M(s, \chi_1) + M(s, \chi_2)).
\]
Now
\[
a(p)\chi_1(p) = \frac{1}{2}(1+i)\chi_1(p) + \frac{1}{2}(1-i)\chi_2(p)
\]
and these are conjugate since \( \chi_1^2 = \chi_2 \) and \( \chi_2^2 = \chi_1 \). Hence
\[
M(s, \chi_1) \text{ and } M(s, \chi_2)
\]
are conjugate for real \( s \), and \( N(s) \) is real.

Let \( s \) be real, greater than 1, and \( \sigma = 1 \). Then
\[
\log M(s, \chi_1) = \sum_p \frac{a(p)\chi_1(p)}{p^s} + O(1)
\]
and
\[
\log M(s, \chi_2) = \sum_p \frac{a(p)\chi_2(p)}{p^s} + O(1).
\]
Now
\[
\sum_p \frac{a(p)\chi_1(p)}{p^s} = \sum_p \frac{a(p)\chi_2(p)}{p^s} = \log L(s; \chi),
\]
\[
\sum_p \frac{a(p)\chi_2(p)}{p^s} = \sum_p \frac{a(p)\chi_1(p)}{p^s} = \log L(s; \chi) + O(1),
\]
Hence
\[
\log M(s, \chi_1) = (1 - i)\log \frac{1}{2} - O(1),
\]
\[
N(s) = \Re M(s, \chi) = \frac{1}{2}\log \frac{1}{2} + O(1).
\]
It is clear from this formula that \( N(s) \) has a zero at each of the points
\[
s = 1 + \frac{1}{4}\pi \text{ near } \{m = 1, 2, \ldots\}.
\]
The method is due to Davenport and Heilbronn (1), (3); they proved that a class of functions, of which an example is
\[ \sum_{n \leq x} \frac{1}{n} \sum_{d|n} \mu(d) \frac{1}{d} \log \frac{T}{4\pi d} \]
has an infinity of zeros for \( \sigma > 1 \). It has been shown by calculation\(^\dagger\) that this particular function has a zero in the critical strip, not on the critical line. The method throws no light on the general question of the occurrence of zeros of such functions in the critical strip, but not on the critical line.

NOTES FOR CHAPTER 10

10.25. In §10.1 Titchmarsh's comment on Riemann's statement about the approximate formula for \( N(T) \) is erroneous. It is clear that Riemann meant that the relative error \( \left| N(T) - L(T) \right| / N(T) \) is \( O(T^{-1}) \).

10.27. Further work has done on the problem mentioned at the end of §10.25. Davenport and Heilbronn (1), (3) showed in general that if \( Q \) is any positive definite integral quadratic form of discriminant \( d \), such that the class number \( N(d) \) is greater than 1, then the Epstein Zeta-function
\[ \zeta_Q(s) = \sum_{Q(x^2 + y^2) > 0} x^{-s} \quad (\sigma > 1) \]
has zeros to the right of \( \sigma = 1 \). In fact they showed that the number of such zeros up to height \( T \) is at least of order \( T \) (and hence of exact order \( T \)). This result has been extended to the critical strip by Voronin (4), who proved that, for such functions \( \zeta_Q(s) \), the number of zeros up to height \( T \), for \( 1 < \sigma < 1 \), is also of order at least \( T \) (and hence of exact order \( T \)). This answers the question raised by Titchmarsh at the end of §10.25.

10.28. Much the most significant result on \( N_\psi(T) \) is due to Levinson (2), who showed that
\[ N_\psi(T) \geq aN(T) \quad (10.28.1) \]
for large enough \( T \), with \( a = 0.312 \). The underlying idea is to relate the distribution of zeros of \( \zeta(s) \) to that of the zeros of \( \zeta'(s) \). To put matters in
\(^\dagger\) Potter and Davenport (1).

10.29. Their proper perspective we first note that Berndt (1) has shown that
\[ \quad \text{for } \sigma = \sigma_0 + it \quad 0 < t < T, \quad \zeta(s) = 0 \quad \Rightarrow \quad \frac{T}{2\pi} \log \frac{T}{4\pi} + O(\log T), \]
and that Speiser (1) has proved that the Riemann Hypothesis is equivalent to the non-vanishing of \( \zeta(s) \) for \( 0 < \sigma < \frac{1}{2} \). This latter result is related to the unconditional estimate
\[ \quad \text{for } \sigma = \sigma_0 + it \quad 0 < \sigma < \frac{1}{2}, \quad T_1 < t < T_2, \quad \zeta(s) = 0 \quad \Rightarrow \quad \frac{T}{2\pi} \log \frac{T}{4\pi} + O(\log T), \]
zeros being counted according to multiplicity. This is due to Levinson and Montgomery (1), who also gave a number of other interesting results on the distribution of the zeros of \( \zeta(s) \).

We sketch the proof of (10.29.2). We shall make frequent reference to the logarithmic derivative of the functional equation (2.5.4), which we write in the form
\[ \frac{\zeta'(s)}{\zeta(s)} \frac{\zeta(1-s)}{\zeta(1-s)} = -\log \pi - \frac{1}{2} \left( \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} - \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} \right) \]
\[ = -F(s), \quad (10.29.3) \]
so. We note that \( F(\frac{1}{2} + it) \) is always real, and that
\[ \partial \zeta(s)/\partial t = \log(1/2\pi) + t\log t \quad (10.29.4) \]
uniformly for \( t \geq 1 \) and \( |s| = \frac{1}{2} \). To prove (10.29.2) it suffices to consider the case in which the numbers \( T_r \) are chosen so that \( \zeta(s) \) and \( \zeta(1-s) \) do not vanish for \( t = T_r, -1 < \sigma < \frac{1}{2} \). We examine the change in argument in \( \zeta(s)/\zeta(1-s) \) around the rectangle with vertices \( 1/2 - \delta + iT_r, 1/2 - \delta + iT_r, 1/2 + \delta + IT_r, \) and \(-1 - IT_r, \) where \( \delta \) is a small positive number. Along the horizontal sides we apply the ideas of §9.4 to \( \zeta(0) \) and \( \zeta(s) \) separately. We note that \( \zeta(s) \) and \( \zeta(1-s) \) are each \( O(t) \) for \( -3 < \sigma < 1 \). Moreover we also have \( \Im(-1 + it) = \frac{T}{2\pi} \log T \), by the functional equation, and hence also
\[ \zeta(s) \zeta(1-s) = \frac{T}{(1-1 + it)} \frac{T}{(1-1 + it)} \frac{T}{\log T_r}, \]
by (10.28.2) and (10.28.4). The method of §9.4 therefore shows that \( \arg(\zeta(s)) \) and \( \arg(\zeta(1-s)) \) both vary by \( O(\log T_r) \) on the horizontal sides of the
rectangle. On the vertical side \( \sigma = -1 \) we have

\[
\zeta'(\sigma) \frac{\zeta(\sigma)}{\zeta(1+\sigma)} = \log \left( \frac{t}{2\pi} \right) + O(1),
\]

by (10.28.3) and (10.28.4), so that the contribution to the total change in argument is \( O(1) \). For the vertical side \( \sigma = \frac{1}{2} - \delta \) we first observe from (10.28.3) and (10.28.6) that

\[
R \left( \frac{\zeta'(-\delta + i\theta)}{\zeta(1-\delta + i\theta)} \right) > 1
\]

(10.28.5)

if \( t > T \), with \( T \) sufficiently large. It follows that

\[
R \left( \frac{\zeta'(-\frac{1}{2} - \delta + i\theta)}{\zeta(1-\frac{1}{2} - \delta + i\theta)} \right) > 1
\]

(10.28.8)

for \( T_1 < t < T_2 \), if \( \delta = h(T_2) \) is small enough. To see this, it suffices to examine a neighbourhood of a zero \( \rho = \frac{1}{2} + iy \) of \( \zeta(\sigma) \). Then

\[
\zeta'(\rho) = \frac{m}{s-\rho} - m' + O(|s-\rho|),
\]

where \( m > 1 \) is the multiplicity of \( \rho \). The choice \( s = \frac{1}{2} + it \) with \( t \to y \) therefore yields \( R(m') > 1 \), by (10.28.9). Hence, on taking \( s = \frac{1}{2} - \frac{1}{2} + it \), we find that

\[
R \left( \frac{\zeta'(s)}{\zeta(s)} \right) = \frac{m^2}{(s-\rho)^2} + R(m') + O(|s-\rho|) > 1
\]

(10.28.6)

for \( |s-\rho| \) small enough. The inequality (10.28.6) now follows. We therefore see that \( \zeta'(\rho)/\zeta(\rho) \) varies by \( O(1) \) on the vertical side \( \sigma = \frac{1}{2} - \delta \) of our rectangle, which completes the proof of (10.28.3). If we write \( N \) for the quantity on the left of (10.28.2) it follows that

\[
N(T_2) - N(T_1) = \frac{1}{2} \left( N(T_2) - N(T_1) \right) + 2N + O(\log T_2),
\]

(10.28.7)

so that we now require an upper bound for \( N \). This is achieved by applying the 'mellifier' method of (9.23–24 to \( \zeta(1-s) \)). Let \( \eta(T_1, T_2) \) denote the number of zeros of \( \zeta(1-s) \) in the rectangle \( \sigma \in \operatorname{Re}(s) < 2, \ T_1 < \Re(s) < T_2 \). The method produces an upper bound for

\[
\int_{T_1}^{T_2} \frac{1}{2} \left( N(T_2) - N(T_1) \right) \, ds
\]

(10.28.8)

which in turn yields an estimate \( N \leq c \left( N(T_2) - N(T_1) \right) \) for large \( T_2 \). The constant \( c \) in this latter bound has to be calculated explicitly, and must be less than \( \frac{1}{2} \) for (10.28.7) to be of use. This is in contrast to (9.20.5), in which the implied constant was not calculated explicitly, and would have been relatively large. It is difficult to have much hope in advance for how large the constant \( c \) produced by the method will be. The following very loose argument gives one some hope that \( c \) will turn out to be reasonably small, and so it transpires in practice.

In using (10.28.9) to obtain a bound for \( N \) we shall take

\[
u = \frac{1}{2} - a \log T_2,
\]

where \( a \) is a positive constant to be chosen later. The zeros \( \rho = \frac{1}{2} + iy \) of \( \zeta(1-s) \) have an asymmetrical distribution about the critical line. Indeed Levinson and Montgomery [1] showed that

\[
\sum_{r < \gamma < ψ} (1 - r) \frac{T}{2\pi} \log \log T
\]

whence \( \frac{r}{2} - (\log \log \gamma)/\log r \) on average. Thus one might reasonably hope that a fair proportion of such zeros have \( \rho < a \), thereby making the integral (10.28.8) rather small.

We now look in more detail at the method. In the first place, it is convenient to replace \( \zeta(1-s) \) by

\[
\zeta(\sigma) + \frac{\zeta(\sigma)}{\rho(\sigma)} = G(\sigma),
\]

say. If we write \( R(\sigma) = e^{-\gamma/2} \Gamma(\sigma) \) (then (10.28.9), together with the functional equation (2.6.4), yields

\[
\zeta(1-s) = - \frac{R(\sigma) \Gamma(s) \zeta(s)}{\Gamma(1-s)},
\]

so that \( G(\sigma) \) and \( \zeta(1-\sigma) \) have the same zeros for \( t \) large enough. Now let

\[
\psi(\sigma) = \sum_{\sigma < r < \gamma} 1, \tag{10.28.9}
\]

be a suitable 'mellifier' for \( G(\sigma) \), and apply Littlewood's formula (9.9.3) to the function \( G(\sigma) \psi(\sigma) \) and the rectangle with vertices \( u + iT_1, 2 + iT_1, 2 + iT_2, u + iT_2 \). Then, as in (9.16), we find that

\[
N < \frac{T_2}{2\pi} \int_{T_1}^{T_2} \frac{\zeta(s) \psi(s) \, ds}{\zeta(1-s) \psi(1-s)}
\]

\[
< \frac{T_2}{2\pi} \int_{T_1}^{T_2} \log |G(u + it)\psi(u + it)| \, dt + O(\log T_2).
\]
Moreover, as in §9.16 we have
\[ \int_{T_1}^{T_2} \frac{1}{3} (T_j - T_1) \log \left( \int_{T_1}^{T_2} |G(u + it)|^2 dt \right) \]
\[ \leq \int_{T_1}^{T_2} |G(u + it)|^2 dt \sim c(a)(T_2 - T_1). \]

Hence, if we can show that
\[ \int_{T_1}^{T_2} |G(u + it)|^2 dt \sim c(a)(T_2 - T_1) \]
for suitable \( T_1, T_2 \), we will have
\[ N \leq \left( \frac{\log c(a)}{2a} + \alpha(1) \right) (N(T_2) - N(T_1)), \]
where
\[ N(T_2) - N(T_1) \gtrsim \left( 1 - \frac{\log c(a)}{a} + \alpha(1) \right) (N(T_2) - N(T_1)) \]
by (10.28.7).

The computation of the mean value (10.28.10) is the most awkward part of Levinson's argument. In (2) he takes \( y = T_1^{1/4} \) and
\[ b_\alpha = \rho(a) n^{-1} \frac{1}{y} c(a) \]
This leads eventually to (10.28.10) with
\[ c(a) \sim c_0 \left( \frac{1}{24a} - \frac{1}{24a^2} + \frac{1}{24a^3} - \frac{1}{24a^4} \right), \]
The optimal choice of \( a \) is roughly \( a = 1.3 \), which produces (10.28.1) with
\[ \alpha = 0.342. \]

The method has been improved slightly by Levinson [4], [5], Lou [1] and Conrey [1] and the best constant thus far is \( \alpha = 0.3658 \) (Conrey [1]). The principal restriction on the method is that on the size of \( y \) in (10.28.9). The above authors all take \( y = T_1^{1/4} \), but there is some scope for improvement via the ideas used in the mean-value theorems (7.24.6), (7.24.7), and (7.24.7).

10.29. An examination of the argument just given reveals that the right-hand side of (10.28.11) gives an upper bound for \( N^* + N^0 \), where
\[ N^* = \{ s = \frac{1}{2} + it : T_1 < t < T_2, \zeta(s) = 0 \}. \]
(\subsection*{zeros being counted according to multiplicity.} However it is clear from (10.28.8) and (10.28.9) that \( \zeta(s) \) can only vanish if \( \zeta(1/2 + it) \) does. Consequently, if we write \( N^0 \) for the number of zeros of \( \zeta(s) \) of multiplicity \( r \), on the line segment \( s = \frac{1}{2} + it \), \( T_1 < t < T_2 \), we will have
\[ N^* = \sum_{r=1}^{\infty} (r-1)N^0. \]

Thus (10.28.7) may be replaced by
\[ N^0 - \sum_{r=1}^{\infty} (r-2)N^0 = (N(T_2) - N(T_1)) - 2N + N^* + O(\log T). \]
If we now define \( N^0(T) \) in analogy to \( N^0 \), but counting zeros \( \frac{1}{2} + it \) with \( 0 < t < T \), we may deduce that
\[ N^0(T) - \sum_{r=1}^{\infty} (r-2)N^0(T) = N(T), \]
for large enough \( T \), and \( \alpha = 0.342. \) In particular at least a third of the non-trivial zeros of \( \zeta(s) \) not only lie on the critical line, but are simple. This observation is due independently to Heath-Brown [3] and Selberg (unpublished). The improved constants \( \alpha \) mentioned above do not all allow this refinement. However it has been shown by Anderson [4] that (10.29.1) holds with \( \alpha = 0.3532. \)

10.30. Levinson's method can be applied equally to the derivatives \( \zeta^n(s) \) of the function \( \zeta(s) \) given by (2.1.12). One can show that the zeros of these functions lie in the critical strip and that the number of them, \( N_n(T) \) say, for \( 0 < t < T \), is \( N(T) + O_\epsilon(\log T) \). If the Riemann hypothesis holds then all these zeros must lie on the critical line. Thus it is of some interest to give unconditional estimates for
\[ \lim \inf N_n(T), \quad \zeta(s) \quad 0 < t < T, \quad \zeta^n(1/2 + it) = 0 \]
say. Levinson [3, 5] showed that \( a_1 \geq 0.71 \), and Conrey [1] improved and extended the method to give \( a_1 \geq 0.8837, a_2 \geq 0.9664 \) and in general \( a_n = 1 + O(n^{-1}). \)
11.1. In the previous chapters we have been concerned almost entirely with the modulus of \( \tau(x) \), and the various values, particularly zero, which it takes. We now consider the problem of \( \zeta(s) \) itself, and the values of \( s \) for which it takes any given value \( n \).

One method of dealing with this problem is to connect it with the famous theorem of Picard on functions which do not take certain values. We use the following theorem 2:

If \( f(z) \) is regular and never 0 or 1 in \( |z| < 1 \), and \( |f(a)| = n \), then \( |f(\alpha)| \leq \beta \) for \( |\alpha - \alpha_0| < \beta \), where \( 0 < \beta < 1 \).

From this we deduce

**Theorem 11.1.** \( \zeta(s) \) takes every value, with one possible exception, an infinity of times in any strip \( 1 - \beta < s < 1 + \beta \).

Suppose, on the contrary, that \( \zeta(s) \) takes the distinct values \( a \) and \( b \) only a finite number of times in the strip, and so never above \( s = \frac{1}{2} \). Let \( \alpha > \frac{1}{2} \), and consider the function \( f(z) = (z-a)/(b-a) \) in the circle \( C \), of radius \( R > \frac{1}{2} \), and common centre \( \alpha_0 = \frac{1}{2} + \frac{1}{2}i\alpha \). Then

\[ |f(z)| < A(\alpha) \text{ in } C, \text{ and so } |(a-b)| < A(\alpha) \text{ for } 1 < \alpha < \alpha_0, \alpha_0 > \frac{1}{2}. \]

Hence \( \zeta(s) \) is bounded for \( s > 1 \), which is false, by Theorem 11.1. (a).

This proves the theorem.

We should, of course, expect the exceptional value to be zero.

If we assume the Riemann hypothesis, we can use a similar method inside the critical strip; but more detailed results independent of the Riemann hypothesis can be obtained by the method of Diophantine approximation. We devote the rest of the chapter to developments of this method.

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11.2. We restrict ourselves in the first place to the half-plane \( s > 1 \), and we consider, not \( \zeta(s) \) itself, but \( \log \zeta(s) \), via the function defined for \( s > 1 \) by the series

\[ \log \zeta(s) = -\sum_{p} p^{-s} + \frac{1}{2} p^{-2s} + \cdots. \]

We consider at the same time the function

\[ \frac{\zeta(s)}{\zeta(1)} = \sum p^{-s} + \frac{1}{2} p^{-2s} + \cdots. \]

We observe that both functions are represented by Dirichlet series, absolutely convergent for \( s > 1 \), and capable of being written in the form

\[ F(s) = \sum f_p(p^{-s}) + f_p(p^{-2s}) + \cdots, \]

where \( f_p(s) \) is a power-series in \( x \) whose coefficients do not depend on \( x \). In fact

\[ f_p(s) = -\log(1-x), \quad f_p(s) = -\log(1-x) \]

in the above two cases. In what follows \( F(s) \) denotes either of the two functions.

11.3. We consider first the values which \( F(s) \) takes on the line \( c_\alpha \), where \( c_\alpha \) is an arbitrary number greater than 1. On this line

\[ F(s) = \sum f_p(p^{-s} - \zeta(s)), \]

and, as \( t \) varies, the arguments \(-\log p - x \) are, of course, all related. But we shall see that there is an intimate connexion between the set \( \mathcal{U} \) of values assumed by \( F(s) \) on \( c_\alpha \) and the set \( \mathcal{V} \) of values assumed by the functions

\[ \Phi(\epsilon_1, \epsilon_2, \epsilon_3, \ldots) = \sum f_p(p^{-s} - \zeta(s)) \]

of an infinite number of independent real variables \( \epsilon_1, \epsilon_2, \ldots \). We shall in fact show that the set \( \mathcal{U} \), which is obviously contained in \( \mathcal{V} \), is everywhere dense in \( \mathcal{V} \), i.e., that every element of \( \mathcal{V} \) (i.e., every given set of values \( \epsilon_1, \epsilon_2, \ldots \)) and every positive \( \epsilon \), there exists an \( s \) such that

\[ |F(s) - \epsilon| < \epsilon. \]

Since the Dirichlet series from which we start is absolutely convergent for \( s = c_\alpha \), it is obvious that we can find \( N - N(c_\alpha, \epsilon) \) such that

\[ |I = \sum_{n=1}^\infty f_p(\zeta(s) - \epsilon n^\alpha)| \leq \epsilon \]

for any values of the \( \mu_n \), and in particular for \( \mu_n = c_\alpha \), or for

\[ \mu_n = -(\log p)/2n. \]
Now since the numbers $\log p_n$ are linearly independent, we can, by Kronecker's theorem, find a number $s$ and integers $p_1, p_2, \ldots, p_n$ such that

$$-\log p_1 - \log p_2 - \cdots - \log p_n < \eta \quad (n = 1, 2, \ldots, N),$$

$\eta$ being an assigned positive number. Since $F_ \alpha(p_n^{\alpha_1 + \cdots + \alpha_r})$ is, for each $\alpha$, a continuous function of $\theta$, we can suppose $s$ so small that

$$\left| \sum_{z=1}^N \left( F_ \alpha(p_n^{\alpha_1 + \cdots + \alpha_r}) - F_ \alpha(p_n^{\alpha_1 + \cdots + \alpha_r}) \right) \right| < \eta. \quad (11.3.2)$$

The result now follows from (11.3.1) and (11.3.2).

11.4. We next consider the set $W$ of values which $F_ \theta(p)$ takes 'in the immediate neighborhood' of the line $s = \alpha$, i.e. the set of all values of $s$ such that the equation $F_ \theta(p) = w$ has, for every positive $\delta$, a root in the strip $|s - \alpha| < \delta$.

In the first place, it is evident that $U$ is contained in $W$. Further, it is easy to see that $U$ is everywhere dense in $W$. For, for sufficiently small $\delta$ (e.g. for $\delta < \delta_0 - 1$),

$$|F_ \theta(p) - F_ \theta(q)| < \delta_0,$$

for all values of $s$ in the strip $|s - \alpha| < \delta$, so that

$$|F_ \theta(p) - F_ \theta(q)| < \delta_0 \quad (|s - \alpha| < \delta). \quad (11.4.1)$$

Now each value of $u$ is assumed by $F_ \theta(p)$ either on the line $s = \alpha_0$ in which case it is a $s$, or at points $\alpha_1 + \alpha_2$ arbitrarily near the line, in which case, in virtue of (11.4.1), we can find a $s$ such that

$$|s - u| < \delta_0 \quad (|s - \alpha_0| < \delta).$$

We now proceed to prove that $W$ is identical with $V$. Since $U$ is contained in and is everywhere dense in both $V$ and $W$, it follows that each of $V$ and $W$ is everywhere dense in the other. It is therefore obvious that $W$ is contained in $V$, if $V$ is closed.

We shall see presently that much more than this is true, viz. that $V$ consists of all points of an area, indicating the boundary. The following direct proof that $V$ is closed is, however, very instructive.

Let $s^*$ be a limit point of $V$, and let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence of $s$'s tending to $s^*$. To each $\alpha$ there corresponds a point $F_ \theta(p_1, \alpha_2, \alpha_3, \ldots)$ in the space of an infinite number of dimensions defined by $0 < \alpha_1 < 1$ (a = 1, 2, 3, 4), such that $F_ \theta(p_1, \alpha_2, \alpha_3, \ldots) \neq s^* - 1$.

Now since $(F_ \theta(p_1, \alpha_2, \alpha_3, \ldots), \ldots)$ is a bounded set of points (i.e. the coordinates are bounded), it has a limit point $F_ \theta(p_1, \alpha_2, \ldots)$, i.e. a point such that from $(F_ \theta(p_1, \alpha_2, \ldots))$ we can choose a sequence $(p_1, \alpha_2, \ldots)$ such that each coordinate $\alpha_n$ of $F_ \theta(p_1, \alpha_2, \ldots)$ tends to the limit $\alpha_n$ as $r \to \infty$.

11.4 THE VALUES OF $(e)$

It is now easy to prove that $e^*$ corresponds to $s^*$, i.e. that

$$e^* = \Phi_\alpha(p_1, \alpha_2, \ldots),$$

so that $s^*$ is a point of $V$. For the series for $e^*$, viz.

$$\sum_{n=1}^N f_n(p_n^{\alpha_1 + \cdots + \alpha_r}) = \Phi_\alpha(p_1, \alpha_2, \ldots),$$

is uniformly convergent with respect to $r$, since (by Weierstrass's $M$-test) it is uniformly convergent with respect to all the $\alpha_1$, the $n$th term tends to $f_n(p_n^{\alpha_1 + \cdots + \alpha_r})$ as $r \to \infty$. Hence

$$s^* = \lim_{n \to \infty} \sum_{n=1}^N f_n(p_n^{\alpha_1 + \cdots + \alpha_r}) = \Phi_\alpha(p_1, \alpha_2, \ldots),$$

which proves our results.

To establish the identity of $V$ and $W$ it remains to prove that $V$ is contained in $W$. It is obviously sufficient (and also necessary) for this that $W$ should be closed. But that $W$ is closed does not follow, as might perhaps be supposed, from the mere fact that $W$ is the set of values taken by a bounded analytic function in the immediate neighborhood of a line. Thus $e^*$ is bounded and arbitrarily near to 0 in every strip including the real axis, but never actually assumes the value 0. The fact that $W$ is closed (which we shall not prove directly) depends on the special nature of the function $F_ \theta(p)$.

Let $v = \Phi_\alpha(p_1, \alpha_2, \ldots)$ be an arbitrary value contained in $V$. We have to show that $v$ is a member of $W$, i.e. that, in every strip

$$|s - \alpha| < \delta,$

$s^*$ assumes the value $v$.

Let

$$G_\delta(p) = \sum_{n=1}^N f_n(p_n^{\alpha_1 + \cdots + \alpha_r})$$

so that $G_\delta(p) = v$. We choose a small circle $C$ with center $\alpha_0$ and radius less than $\delta$ such that $G_\delta(p) = v$ on the circumference. Let $m$ be the minimum of $|G_\delta(p) - v|$ on $C$.

Kronecker's theorem enables us to choose $\alpha$ such that, for every $s$ in $C$

$$|F_ \theta(p) - G_\delta(p)| < G_\delta(p) \leq m.$$

The proof is almost exactly the same as that used to show that $U$ is everywhere dense in $V$. The series for $F_ \theta(p)$ and $G_\delta(p)$ are uniformly convergent in the strip, and, for each point $N$, $\sum_{n=1}^N f_n(p_n^{\alpha_1 + \cdots + \alpha_r})$ is a continuous function of $\alpha_1, \alpha_2, \ldots, \alpha_r$. It is therefore sufficient to show that we can choose $\alpha$ so that the difference between the arguments of $p_1^{\alpha_1}$ at $s = \alpha_0 + i\delta$ and $p_1^{\alpha_1}$ at $s = \alpha_0$ and consequently that
between the respective arguments at every pair of corresponding points of the two circles is (mod 2π) arbitrarily small for n → 1, 2, ..., N. The possibility of this choice follows at once from Kroncker's theorem.

We now have

\[ F(s + i\theta) = (\phi(s) - \phi) F(s + i\theta_0) - \phi(s) \]

and on the circumference of C

\[ |F(s + i\theta) - \phi(s)| < \rho \leq |\phi(s)| \]

Hence, by Rouche's theorem, \( F(s + i\theta) - \phi \) has in C the same number of zeros as \( \phi(s) - \phi \), and so at least one. This proves the theorem.

11.5. We now proceed to the study of the set \( V_\rho \). Let \( V_{\rho_n} \) be the set of values taken by \( f_n(p_{\rho_n}) \) for \( \rho = \rho_n \), i.e. the set taken by \( f_n(s) \) for \( \psi = \rho \). Then \( V_\rho \) is the 'sum' of the sets of points \( V_{\rho_n} \), i.e. it is the set of all values \( \rho_1 + \rho_2 + ... \), where \( \rho_1 \) is any point of \( V_{\rho_1} \), \( \rho_2 \) any point of \( V_{\rho_2} \), and so on. For the function \( \log(\zeta(s)) \), \( V_\rho \) consists of the points of the curve described by \( -\log(1-x) \) as \( x \) describes the circle \( |x| = \rho \), for \( \zeta(s)/\zeta(0) \) it consists of the points of the curve described by \( -\log(p_\rho)/(1-x) \).

We begin by considering the function \( \zeta(s)/\zeta(0) \). In this case we can find the set \( V \) explicitly. Let

\[ w_\rho = -\frac{\log(p_\rho)}{\log(1-z)} \]

As \( z_\rho \) describes the circle \( |z| = \rho \), \( w_\rho \) describes the circle with centre

\[ c_\rho = -\frac{\rho \log(p_\rho)}{1-\rho} \]

and radius

\[ \rho_\rho = \frac{\rho \log(p_\rho)}{1-\rho} \]

and let

\[ w_\rho = c_\rho + w_\rho \]

and let

\[ c = \sum_{\rho_\rho} c_\rho = \frac{1}{\zeta(0)} \]

Then \( V \) is the set of all the values of

\[ z = \sum_{\rho_\rho} \rho_\rho z_\rho \]

for independent \( \phi_1, \phi_2, ... \). The set \( V \) of the values of \( \sum \rho_\rho z_\rho \) is the 'sum' of an infinite number of circles with centre at the origin, whose radii \( \rho_1, \rho_2, ... \) form, as it is easy to see, a decreasing sequence. Let \( V_\rho \) denote the nth circle.

11.5. THE VALUE OF \( \zeta(0) \)

Then \( V_1 + V_2 \) is the area swept out by the circle of radius \( \rho_1 \) as its centre describes the circle with centre the origin and radius \( \rho_1 \). Hence, since \( \rho_1 < \rho_1 \), \( V_1 + V_2 \) is the annulus with radii \( \rho_1 - \rho_2 \) and \( \rho_1 + \rho_2 \).

The argument clearly extends to any finite number of terms. Thus \( V_1 + ... + V_N \) consists of all points of the annulus

\[ \rho_1 - \frac{\sum_1^N \rho_n}{\sum_1^N \rho_n} \leq |\omega| \leq \frac{\sum_1^N \rho_n}{\sum_1^N \rho_n} \]

or, if the left-hand is negative, of the circle

\[ |\omega| \leq \frac{\sum_1^N \rho_n}{\sum_1^N \rho_n} \]

It is now easy to see that

(i) \( \frac{\rho_1}{\sum_1^N \rho_n} \) lies within V.

(ii) \( \frac{\rho_1}{\sum_1^N \rho_n} \) lies outside V.

For example, in case (ii), let \( w_\rho \) be an interior point of the circle. Then we can choose \( N \) so large that

\[ \sum_{\rho_\rho} \rho_\rho < \sum_{\rho_\rho} \rho_\rho - |w_\rho| \]

and

\[ w_\rho = \sum_{\rho_\rho} \rho_\rho \]

lies within the circle \( V_1 + ... + V_N \) for any values of the \( \phi_n \), e.g. for \( \phi_{n+1} = ... = 0 \). Hence

\[ w_\rho = \frac{\sum_1^N \rho_n}{\sum_1^N \rho_n} \]

for some values of \( \phi_n, ... \phi_1 \), and so

\[ w_\rho = \sum_1^N \rho_n \]

as required. That \( V \) also includes the boundary in each case is clear on taking all the \( \phi_n \) equal.

The complete result is that there is an absolute constant \( D = 2.07 \), determined as the root of the equation

\[ \frac{2-\log 2}{1-2^{3/2}} = \frac{\sum_1^N \rho_n}{\sum_1^N \rho_n} \]

\[ 1-\rho_1 = \frac{\rho_1 \rho_2}{\sum_1^N \rho_n} \].
such that for $c_0 > D$ we are in case (i), and for $1 < c_0 < D$ we are in case (ii). The radius of the outer boundary of $S$ is

$$R = \zeta(2c_0)/\zeta(c_0)$$

in each case; the radius of the inner boundary in case (i) is

$$r = 2c_0 - R = 2^{1-c_0}\log(1-2^{-c_0}) - R.$$ 

Summing up, we have the following results for $\zeta(c)/\zeta(2c)$.

**Theorem 11.5 (A).** The values which $\zeta(c)/\zeta(2c)$ takes on the line $s = c_0 > 1$ form a set everywhere dense in a region $R(c_0)$. If $a_0 > D$, $R(a_0)$ is the annulus (boundary included) with centre $c$ and radii $r$ and $R$; and if $a_0 < D$, $R(a_0)$ is the circular area (boundary included) with centre $c$ and radius $R$; $c$, $r$, and $R$ are continuous functions of $a_0$ defined by

$$c = \zeta(2a_0)/\zeta(a_0), \quad R = c - \zeta(c)/\zeta(2c), \quad r = 2^{-c_0}\log(1-2^{-c_0}) - R.$$

Further, as $a_0 \to \infty$, $\lim c = \lim r = \lim R = 0$, $\lim c/R = \lim (R-r)/R = 0$, $\lim c_0 = \lim r = 0$, and as $a_0 \to 1$, $\lim R = \infty$, $\lim c = \zeta(2)/\zeta(2)$.

**Theorem 11.5 (B).** The set of values which $\zeta(s)/\zeta(2s)$ takes in the immediate neighbourhood of $s = a_0$, is identical with $R(a_0)$. In particular, since $c$ tends to a finite limit $R$ and $R$ to infinity as $a_0 \to 1$, $\zeta(s)/\zeta(2s)$ takes all values infinitely often in the strip $1 s < 1 + \delta$, for an arbitrary positive $\delta$.

The above results evidently enable us to study the set of points at which $\zeta(c)/\zeta(2c)$ takes the assigned value $s$. We confine ourselves to giving the result for $a = 0$; this is the most interesting case, since the zeros of $\zeta(c)/\zeta(2c)$ are identical with those of $\zeta(s)$.

**Theorem 11.5 (C).** There is an absolute constant $E$, between 2 and 3, such that $\zeta(c)$ is $0$ for $a > E$, while $\zeta(2c)$ has an infinity of zeros in every strip between $a = 1$ and $a = E$.

In fact it is easily verified that the annulus $R(c_0)$ includes the origin if $a_0 = 2$, but not if $a_0 = 3$.

11.6. We proceed now to the study of $\log \zeta(s)$. In this case the set $V_s$ consists of the 'sum' of the curves $V_s$ described by the points

$$u_s = -\log(1-t_s)$$

as $t_s$ describes the circle $|t_s| = p_0^{-2s}$. In the first place, $V_s$ is a convex curve. For if

$$u + iv = w = f(t) = f(u+iv),$$

the curve $V_s$ describes a convex curve with centre the origin. Let $V'_s$ be the 'sum' of these figures.

Hence the curve $V'_s$ is therefore convex and symmetrical about the line $u = 0$, and $u = v = 0$.

The curve $V'_s$ is therefore convex and symmetrical about the line

$$u = \frac{1}{2}\log((1+r)(1-r))$$

and $u = v = 0$. Its diameter in the $u$ and $v$ directions are $\frac{1}{2}\log((1+r)(1-r))$ and $\lim c_0 = \log P_0$. Then the points $u_s$ describe symmetrical convex figures with centre the origin. Let $V'_s$ be the 'sum' of these figures.

It is now easy, by analogy with the previous case, to imagine the result. The set $V'_s$, which is plainly symmetrical about both axes, is either (i) the region bounded by two convex curves, one of which is entirely interior to the other, or (ii) the region bounded by a single convex curve. In each case the boundary is included as part of the region.

This follows from a general theorem of Bohr on the 'summation' of a series of convex curves.
For our present purpose the following weaker but more obvious results will be sufficient. The set $V'$ is included in the circle with centre the origin and radius

$$R = \sum_{n=1}^{\infty} \frac{\log \frac{1+\delta_n}{1-\delta_n}}{\delta_n} = \log \frac{\delta_1}{1-\delta_1}.$$  

If $\delta_n$ is sufficiently large, $V'$ lies entirely outside the circle of radius

$$a \sin 2^n > \sum_{n=1}^{\infty} \frac{1}{\delta_n} \frac{1+\delta_n}{1-\delta_n} \sin 2^n + \frac{1}{2} \log \frac{1+\delta_n}{1-\delta_n} - R. $$

If

$$\sum_{n=2}^{\infty} \sin 2^n \delta_n > \frac{1}{2} \log \frac{1+\delta_1}{1-\delta_1}$$

and so if $\delta_n$ is sufficiently near to 1, $V'$ includes all points inside the circle of radius

$$\sum_{n=2}^{\infty} \sin 2^n \delta_n.$$

In particular $V'$ includes any given area, however large, if $\delta_n$ is sufficiently near to 1.

We cannot, as in the case of circles, determine in a satisfactory manner whether we are in case (i) or case (ii). It is not obvious, for example, whether there exists an absolute constant $D'$ such that we are in case (i) or (ii) according as $\delta_n > D'$ or $1 < \delta_n < D'$. The discussion of this point demands a closer investigation of the geometry of the special curves with which we are dealing, and the question would appear to be one of considerable interest.

The relations between $U$, $V$, and $W$ will now give us the following analogues for $\log \phi(x)$ of the results for $\log \phi(\theta)/\phi(\theta)$.

**Theorem 11.6 (A).** On each line $a = a_0 > 1$ the values of $\log \phi(x)$ are everywhere dense in a region $R(a)$ which is either (i) the ring shaped area bounded by two convex curves, or (ii) the area bounded by one convex curve. For sufficiently large values of $a_0$ we are in case (i), and for values of $a_0$ sufficiently near to 1 we are in case (ii).

**Theorem 11.6 (B).** The set of values which $\log \phi(x)$ takes in the immediate neighborhood of $a = a_0$ is identical with $R(a_0)$. In particular, since $R(a)$ includes any given finite area when $a_0$ is sufficiently near 1, $\log \phi(x)$ takes every value in this area of $1 < a < 1 + \delta$.

As a consequence of the last result, we have

**Theorem 11.6 (C).** The function $\phi(x)$ takes every value except 0 an infinity of times in the strip $1 < a < 1 + \delta$.

This is a more precise form of Theorem 11.1.

11.7. We have seen above that $\log \phi(x)$ takes any assigned value $a$ an infinity of times as $a \to 1$. It is natural to raise the question how often the value $a$ is taken, i.e. the question of the behaviour for large $T$ of the number $M(T)$ of roots of $\log \phi(x) = a$ in $a > 1, 0 < t < T$. This question is evidently closely related to the question as to how often, as $t \to \infty$, the point $(a_1, a_2, \ldots, a_n)$ of the plane of the theorem, comes (mod 1) arbitrarily near every point in the $N$-dimensional unit cube, some within a given distance of an assigned point $(b_1, b_2, \ldots, b_N)$. The answer to this last question is given by the following theorem, which asserts that, roughly speaking, the point $(a_1, a_2, \ldots, a_N)$ comes near every point of the unit cube equally often, i.e. it does not give a preference to any particular region of the unit cube.

Let $a_1, a_2, \ldots, a_N$ be linearly independent, and let $\gamma$ be a region of the $N$-dimensional unit cube with volume $\Gamma$ (in the Jordan sense). Let $L_n(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point $P(a_1, a_2, \ldots, a_n)$ is (mod 1) included in $\gamma$. Then

$$\lim_{T \to \infty} L_n(T)/\Gamma = 1.$$  

The region $\gamma$ is said to have the volume $\Gamma$ in the Jordan sense, if, given $\epsilon$, we can find two sets of cubes with sides parallel to the axes, of volumes $\Gamma_\epsilon$ and $\Gamma_\epsilon'$, included in $\gamma$ and excluding $\gamma$ respectively, such that

$$\Gamma_\epsilon - \epsilon < \Gamma < \Gamma_\epsilon + \epsilon.$$  

If we call a point with coordinates of the form $(a_1, a_2, \ldots, a_n)$, mod 1, an 'accessible' point, Kronecker's theorem states that the accessible points are everywhere dense in the unit cube $C$. If now we take two equal cubes with sides parallel to the axes, and with centres at accessible points $P_1$ and $P_2$, corresponding to $t_1$ and $t_2$, it is easily seen that

$$\lim_{t_1 \to t_2} L_n(T)/\Gamma_\epsilon = 1.$$  

For $(a_1, a_2, \ldots, a_n)$ will lie inside $\gamma$ when and only when $(a_1(t_1-h_1-t_2), \ldots, a_n(t_1-h_1-t_2))$ lies inside $\gamma$. Consider now a set of $p$ non-overlapping cubes $c$, inside $\gamma$, of side $\epsilon$, each of which has its centre at an accessible point, and $y$ of which lie inside $\gamma$, and a set of $P$ overlapping cubes $c$, also centred on accessible points, whose union includes $\gamma$ and such that $y$ is included in a union of $Q$ of them. Since the accessible points are everywhere dense, it is possible to choose the cubes such that $\partial P$ and $Q\partial P$ are arbitrarily near to $\Gamma$. Now, denoting by $\Sigma L(T)$ the sum of $\Sigma t$ intervals in $(0, T)$ corresponding to the cubes $c$ which lie in $\gamma$, and so on,

$$\Sigma L(T) \leq \Sigma L(T) \leq \frac{1}{2} L(T)/\Sigma L(T).$$
Making $T \to \infty$ we obtain
\[ \frac{q}{p} = \lim_{T \to \infty} \frac{L(t)}{T} = \frac{q}{p}, \]
and the result follows.

**Theorem 11.8 (a).** If $\sigma = \sigma_0 > 1$ is a line on which $\log \xi(t)$ comes arbitrarily near to a given number $a$, then in every strip $\sigma_0 - \delta < \sigma < \sigma_0 + \delta$ the value $a$ is taken more than $K(a, m_0, b)T$ times, for large $T$, in $0 < t < T$.

To prove this we have to reconsider the argument of the previous sections, as well as to the existence of the root of $\log \xi(t) - a$ in the strip, and use Kronecker’s theorem in its generalized form. We shall see that a sufficient condition that $\log \xi(t) - a$ may have a root inside a circle with centre $\sigma_0 + i \mu$ and radius $\delta$ is that, for a certain $N$ and corresponding numbers $\theta_0, \ldots, \theta_N$, and a certain $\eta = \eta(\sigma_0, \delta, \theta_0, \ldots, \theta_N)$
\[ -\eta \log p_{\sigma_0} - 2\eta \log p_{\delta} - 3\eta \mu < 0 \quad (n = 1, 2, \ldots, N). \]

From the generalized Kronecker’s theorem it follows that the sum of the intervals between 0 and $T$ in which $\zeta$ satisfies this condition is asymptotically equal to $(\gamma(2\pi)^{-1})T$, and it is therefore greater than $\frac{1}{2}(\gamma(2\pi)^{-1})T$ for large $T$. Hence we can select more than $\frac{1}{2}(\gamma(2\pi)^{-1})T$ numbers $\zeta$ in them, no two of which differ by less than $\delta$. If now we describe circles with the points $\sigma_0 + i \mu$ as centres and radius $\delta$, these circles will not overlap, and each of them will contain a zero of $\log \xi(t) - a$. This gives the desired result.

We may prove

**Theorem 11.8 (b).** There are positive constants $K_1(a)$ and $K_2(a)$ such that the number $M(T)$ of zeros of $\log \xi(t) - a$ in $\sigma > 1$ satisfies the inequalities
\[ K_1(a)T < M(T) < K_2(a)T. \]

The lower bound follows at once from the above theorem. The upper bound follows from the more general result that if $f$ is any given constant, the number of zeros of $\log \xi(t) - a$ in $\sigma > 1 + \delta$ $(\delta > 0)$, $0 < \sigma < T$, is $O(T)$ as $T \to \infty$.

The proof of this is substantially the same as that of Theorem 9.15 (A), the function $\xi(t) - a$ playing the same part as $\xi(t)$ did there. Finally the number of zeros of $\log \xi(t) - a$ is not greater than the number of zeros of $\xi(t) - a$, and so is $O(T)$.

**Lemma.** If $f(t)$ is regular for $|t - t_0| < R$, and
\[ \int_{-\infty}^{\infty} f(t)^2 dt = H, \]
then
\[ \int_{-\infty}^{\infty} |f(t)|^2 R^{2M} = \int |f(t)|^2 R^{2M} dt. \]

For if $|t - t_0| < R$,
\[ f(t) = \frac{1}{2\pi} \int \frac{f(t_0) - R}{t - t_0} dt_0. \]

Hence
\[ \int f(t)^2 R^{2M} dt = \frac{1}{2\pi} \int \int \frac{f(t_0)^2}{R} dt_0 \int dt. \]

The result follows.

**Theorem 11.9.** Let $a_0$ be a fixed number in the range $\frac{1}{2} < \sigma < 1$. Then the values which $\log \xi(t)$ takes on $\sigma = a_0$, $t > 0$, are everywhere dense in the whole $t$-plane.

Let
\[ \zeta_0(t) = \xi(t) \prod_{\zeta \in \Omega(t)} (1 - p_t^\zeta). \]

This function is similar to the function $\zeta_0(t)M(t)$ of Chapter IX, but it happens to be more convenient here.

Let $\delta$ be a positive number less than $\frac{1}{2}(a_0 - \frac{1}{2})$. Then it is easily seen as in § 9.19 that for $N > N(a_0, \delta)$, $T > T_0 = T_0(N)$,
\[ \int |\zeta_0(t) - 1 - \delta| dt < cT \]
uniformly for $a_0 - \delta < \sigma < a_0 + \delta$ $(a_0 > 1)$. Hence
\[ \int \int |\zeta_0(t) - 1 - \delta| dt_0 dt < (a_0 - a_0 - \delta)(a_0 + \delta). \]

Hence
\[ \int |\zeta_0(t) - 1 - \delta| dt_0 dt < (a_0 - a_0 - \delta)(a_0 + \delta). \]
for more than \((1 - e^{-t})T\) integer values of \(r\). Since this rectangle contains the circle with centre \(x = r + t\), where \(e_0 < r < e_1, x - \frac{1}{2} + \delta < t < x + \frac{1}{2} - \delta\), and radius \(\delta\), it is easily seen from the lemma that we can choose \(\delta\) and \(c\) so that given \(0 < q < 1, 0 < q' < 1\), we have
\[ \|v(x + t) - 1\| < \eta \quad (e_0 < r < e_1) \] (11.9.1)
for a set of values of \(t\) of measure greater than \(1 - q'T\), and for
\[ N \geq N(\eta, \eta') \quad T \geq T_0(N). \]

Let
\[ R_0(x) = \sum_{\nu = 1}^{\infty} \log(1 - p_\nu^* \sigma^2) \quad (\sigma > 1), \]
where \(\log\) denotes the principal value of the logarithm. Then
\[ L_0(x) = \exp(R_0(x)). \]
We want to show that \(B_0(x) = \log L_0(x)\), i.e. that \(|B_0(x)| < \frac{1}{2} r\), for \(\sigma > e_0\) and the values of \(t\) for which (11.9.1) holds. This is true by (11.9.1), \(R_0(x) > 0\) for \(e_0 < \sigma < \sigma_0\), so that \(R_0(x)\) must remain between \(-\frac{1}{2} r\) and \(\frac{1}{2} r\) for all values of \(\sigma\) in this interval. This gives the desired result.

We have therefore
\[ |B_0(x)| = \left| \log(1 - L_0(x)) \right| < 2 \|L_0(x) - 1\| < 2\eta \]
for \(e_0 < \sigma < \sigma_0\). \(N \geq N(\eta, \eta', \eta'')\), \(T \geq T_0(N)\), in a set of values of \(t\) of measure greater than \((1 - \eta')T\).

Now consider the function
\[ F_0(\sigma + t) = -\sum_{\nu = 1}^{\infty} \log(1 - p_\nu^* \sigma^2), \]
and in conjunction with it the function of \(N\) independent variables
\[ \Phi_0(\eta_0, \ldots, \eta_N) = -\sum_{\nu = 1}^{\infty} \log(1 - p_\nu^* \eta_0^N \ldots \eta_N^N). \]
Since \(\sum p_\nu^* \eta^N\) is divergent, it is easily seen from our previous discussion of the values taken by \(\log(\sigma)\) that the set of values of \(\Phi_0\) includes any given finite region of the complex plane if \(N\) is large enough. In particular, if \(\sigma\) is any given number, we can find a number \(N\) and values of the \(\eta's\) such that
\[ \Phi_0(\eta_0, \ldots, \eta_n) = a. \]

We can then, by Kneser's theorem, find a number \(t\) such that \(|F_0(\sigma + t) - a|\) is arbitrarily small. But this in itself is not sufficient to prove the theorem, since this value of \(t\) does not necessarily make \(|B_0(x)|\) small. An additional argument is therefore required.

11.9

THE VALUES OF \((\alpha)\)

Let
\[ \Phi_M(x) = -\sum_{\nu = 1}^{\infty} \log(1 - p_\nu^* \alpha^2 \eta) = \sum_{\nu = 1}^{\infty} \sum_{m = 1}^{M} \frac{p_\nu m \eta^m}{m}. \]
Then, expressing the squared modulus of this as the product of conjugates, and integrating term by term, we obtain
\[ \int_{\frac{1}{2} + \delta}^{T} \frac{1}{2} \sum_{\nu = 1}^{\infty} \frac{1}{m} \sum_{m = 1}^{M} p_\nu m \eta^m \]
\[ < \sum_{\nu = 1}^{\infty} \frac{p_\nu m \eta^m}{m} < \ldots < 4 \]
which can be made arbitrarily small, by choice of \(M\), for all \(N\). It therefore follows from the theory of Kiemann integration of a continuous function that, given \(\epsilon\), we can divide up the \((N - M)\)-dimensional unit cube into sub-cubes \(\xi_\nu\), each of volume \(\lambda\), in such a way that
\[ \lambda \sum_{\nu} \max \Phi_M(x)^2 < 4\epsilon. \]
Hence for \(M > M(\epsilon)\) and any \(N \geq N_0\), we can find cubes of total volume greater than \(\frac{1}{2}\) in which \(\Phi_M(x) < \epsilon\).

We now choose our value of \(t\) as follows.

(i) Choose \(M\) so large, and give \(\theta_0, \ldots, \theta_M\) such values, that
\[ \Phi_M(\theta_0, \ldots, \theta_M) = 0. \]
It then follows from considerations of continuity that, given \(\epsilon\), we can find an \(M\)-dimensional cube with centre \(\theta_0, \ldots, \theta_M\) and side \(a > 0\) throughout which \(\Phi_M(\theta_0, \ldots, \theta_M - a) < \epsilon\).

(ii) We may also suppose that \(M\) has been chosen so large that, for any value of \(N\), \(\Phi_M(x) < \epsilon\) in certain \((N - M)\)-dimensional cubes of total volume greater than \(\frac{1}{2}\).

(iii) Having fixed \(M\) and \(d\), we can choose \(N\) so large that, for \(T > T_0(N)\), the inequality \(|B_0(x)| < \epsilon\) holds in a set of values of \(t\) of measure greater than \((1 - dT)^2\).

(iv) Let \(I(T)\) be the sum of the intervals between \(0\) and \(T\) for which the point
\[ (-\log p_0, \ldots, -\log p_n)/2\]
is (mod 1) inside one of the \(N\)-dimensional cubes, of total volume greater than \(d^N\), determined by the above construction. Then by the extended Kneser's theorem, \(I(T) > d^N\) if \(T\) is large enough. There are
therefore values of $t$ for which the point lies in one of these cubes, and for which at the same time $\left| |B_t(s)| - 1 \right| < \frac{1}{2}$. For such a value of $t$

$$\left| \log (\zeta(s)-a) \right| \leq \left| \zeta(s)-a \right| + \left| |B_t(s)| - 1 \right| + \left| \Phi_M(s) \right| + \left| \Phi(s) \right|$$

$$< 1 + 1 + \frac{1}{2} + \varepsilon = 3$$

and the result follows.

11.10. Theorem 11.10. Let $\frac{1}{2} < a < \beta < 1$, and let $a$ be any complex number. Let $M_0(a, \beta)$ be the number of zeros of $\log (\zeta(s)-a)$ (defined as before) in the rectangle $a < \sigma < \beta$, $0 < t < T$. Then there are positive constants $K_1(a, \alpha, \beta), K_2(a, \alpha, \beta)$ such that

$$K_1(a, \alpha, \beta)T < M_0(a, \beta)T < K_2(a, \alpha, \beta)T \quad (T > T_1).$$

We first observe that, for suitable values of the $\theta$'s, the series

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$. This is true, for example, if $\nu_0 = \frac{1}{4}$ for sufficiently large values of $n$; for then

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}) = \sum_{a < \sigma < \beta} (-1)^n \log (1 - p^{-\sigma} \zeta^{(a+\beta)})$$

is convergent for real $x > 0$, and hence uniformly convergent in any finite region to the right of the imaginary axis; and for any $\sigma$'s

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}) = \sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$. If $\sigma$ is any given number, and the $\theta$'s have that property, we can choose $\nu_0$ so large that

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}) < \varepsilon \quad (\sigma = \frac{1}{2} + \beta),$$

and at the same time so that the set of values of

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)})$$

includes the circle with centre the origin and radius $|a| + |\varepsilon|$. Hence by choosing first $\theta_1, \ldots, \theta_n$, then $\theta_n, \ldots, \theta_1$, we can find values of the $\theta$'s, say $\theta_1, \ldots, \theta_n$, such that the series

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$, and

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}) = \chi.$$

We can then choose a circle $C$ of centre $\frac{1}{2} + \varepsilon$ and radius $\rho < \frac{1}{2}(\beta - \alpha)$ on which $G(s) = \chi$.

Let

$$m = \min_{s \in C} |G(s) - \chi|.$$}

Now let

$$\Phi_M(s) = -\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}).$$

Then, as in the previous proof,

$$\sum_{a < \sigma < \beta} \log (1 - p^{-\sigma} \zeta^{(a+\beta)}) = \chi.$$
It now follows as before that there is a set of values of \( t \) in \( (0, T) \), of measure greater than \( KT \), such that for \( \beta = \beta(T) \),
\[
\sum_{x \leq T} \log(1-p_x^{-\beta \log x}) - \sum_{x \leq T} \log(1-p_x^{-\beta}) < \frac{1}{3} m, \]
and also
\[
|\mathcal{M}_n(a)| < \frac{1}{3} m, \]
and so
\[
|\mathcal{R}_n(a+i\theta)| < \frac{1}{3} m. \]
At the same time we can suppose that \( M \) has been taken so large that
\[
|\overline{\theta}(a) + \sum_{x \leq T} \log(1-p_x^{-\beta \log x})| < \frac{1}{3} m \quad (a > 0) \]
Then
\[
|\log \xi(a) - \overline{\theta}(a)| < \frac{1}{3} m \]
on the circle with centre \(|-a| = \beta + i\theta \) and radius \( \beta \). Hence, as before, \( \log \xi(a) - a\) has at least one zero in such a circle. The number of such circles for \( 0 < \alpha < T \) which do not overlap is plainly greater than \( KT \).
The lower bound for \( M\) follows, the upper bound holds by the same argument as in the case \( \alpha = 0 \).
It has been proved by Bohr and Jensen, by a more detailed study of the situation, that there is a \( K(\alpha, \beta) \) such that
\[
M(\alpha, \beta; T) \sim K(\alpha, \beta) T. \]
An immediate corollary of Theorem 11.10 states, if \( N(\alpha, \beta; T) \) is the number of points in the rectangle \( \frac{1}{2} < a < b < \beta < 1, 0 < \alpha < \epsilon \) where \( \xi(a) = a \) (\( a \neq 0 \)), then
\[
N(\alpha, \beta; T) \geq K(\alpha, \beta) T \quad |T| \geq T_0. \]
For \( \xi(n) = a \) if \( \log \xi(a) = b \log n + \beta \) is the value of the right-hand side being taken. This result, in conjunction with Theorem 9.17, shows that the value \( 0 \) of \( \xi(a) \), if it occurs at all in \( a > \frac{1}{2} \), is at any rate quite exceptional, zeros being infinitely rarer than \( a \) values for any value of \( a \) other than zero.

**NOTES FOR CHAPTER 11**

11.11. Theorem 11.9 has been generalised by Voronin [1], [2], who obtained the following 'universal' property for \( \xi(a) \). Let \( D_\beta \) be the closed disc of radius \( r_\beta \) centred at \( a \), and let \( f(x) \) be any function continuous and non-vanishing on \( D_\beta \), and holomorphic on the interior of \( D_\beta \). Then for any \( \epsilon > 0 \) there is a real number \( t \) such that
\[
\max_{x \in \mathbb{R}} |(\xi(a + i\theta) - f(x))| < \epsilon. \quad (11.11.1) \]

It follows that the curve
\[
\gamma(t) = (\xi(a + i\theta), \xi(a + i\theta), \ldots, \xi^{(n)}(a + i\theta)) \]
is dense in \( C^n \), for any fixed \( n \) in the range \( 1 < n < 1 \) (in fact Voronin [1] establishes this for \( n = 1 \) also). To see this we choose a point \( x = (z_1, z_2, \ldots, z_n) \) with \( z_0 \neq 0 \), and take \( f(z) \) to be a polynomial for which \( f^{(n)}(z) = z_n \), for \( 0 < m < n \). We then fix an \( R \) such that \( 0 < R < \frac{1}{2} \), and such that \( f(x) \) is nonvanishing on the closed disc \( |x - a| < R \). Thus, if \( R = R(\beta - \frac{1}{2}, \epsilon) \), the disc \( D_\beta \) contains the circle \( |z - a| = R \), and hence (11.1.1.1) in conjunction with Cauchy's inequality
\[
|f^{(n)}(z)| < M^n |f(z)| \quad \text{max}_{0 < m < n} \|f(z)\| \]
yields
\[
|f^{(m)}(z)| < M^n \quad \text{max}_{0 < m < n} \|f(z)\| \]
Hence \( \gamma(t) \) comes arbitrarily close to \( z \). The required result then follows, since the available \( z \) are dense in \( C^n \).

Voronin's work has been extended by Bagchi [1] (see also Gonek [1]) so that \( D_\beta \) may be replaced by any compact subset \( D \) of the strip \( a < \beta < 1 \), whose complement in \( C^n \) is connected. The condition on \( f \) is then that it should be continuous and non-vanishing on \( \overline{D} \) and holomorphic on the interior (if any) of \( D \). From this it follows that if \( \Phi \) is any continuous function, and \( h_1 < h_2 < \ldots < h_m \) are real constants, then \( \xi(a) \) cannot satisfy the differential-difference equation
\[
\Phi([\xi(a + h_1), \xi(a + h_2), \ldots, \xi^{(n)}(a + h_1), \xi(a + h_2), \ldots, \xi^{(n)}(a + h_1), \ldots]) = 0 \]
unless \( \Phi \) vanishes identically. This improves earlier results of Ostrowski [1] and Reich [1].

11.12. Levinson [6] has investigated further the distribution of the solutions \( \rho_\xi = \rho_\xi(\xi(a) = a) \). The principal results are that
\[
\# \{x : 0 < \gamma(x) < T \} = \frac{T}{2\pi} |\log T + O(T)| \]
and
\[
\# \{x : 0 < \gamma(x) < T \} = \frac{1}{2\pi} |\log T + O(T)| \quad (\delta > 0) \]
Thus (c.f. § 9.15) all but an infinitesimal proportion of the zeros of \( \xi(a) = a \) lie in the strip \( \frac{1}{2} - \delta < a < \frac{1}{2} + \delta \), however small \( \delta \) may be.
In reviewing this work Montgomery (Math. Reviews 63 #10709) quotes an unpublished result of Selberg, namely
\[
\sum_{\zeta(\frac{1}{2} + it) \neq 1} (\beta_n - 1) - \frac{1}{4\xi} T \log(\log T)^{\frac{1}{4}}.
\] (11.12.1)

This leads to a stronger version of the above principle, in which the infinite strip is replaced by the region
\[
|\sigma - \frac{1}{2}| < \frac{\phi(t) \log(t)}{\log t},
\]
where \(\phi(t)\) is any positive function which tends to infinity with \(t\). It should be noted for comparison with (11.12.1) that the estimate
\[
\sum_{\alpha_n \in R} (\beta_n - 1) = O(\log T)
\]
is implicit in Levinson's work. It need hardly be emphasized that despite this result the numbers \(\beta_n\) are far from being symmetrically distributed about the critical line.

11.13. The problem of the distribution of values of \(\zeta(\frac{1}{2} + it)\) is rather different from that of \(\zeta(\sigma + it)\) with \(\frac{1}{2} < \sigma < 1\). In the first place it is not known whether the values of \(\zeta(\frac{1}{2} + it)\) are everywhere dense, though one would conjecture so. Secondly there is a difference in the rates of growth with respect to \(t\). Thus, for a fixed \(\varepsilon > 0\), Bohr and Jessen (1, 2) have shown that there is a continuous function \(R(\sigma, \varepsilon)\) such that
\[
\frac{1}{2\pi} \cdot m \{ t \in [-T, T] : \log \zeta(\sigma + it) \in R \} \cdot \int_{-\infty}^{\infty} F(x + iy; \sigma) \, dx \, dy \quad (T \to \infty)
\]
for any rectangle \(R \subset C\) whose sides are parallel to the real and imaginary axes. Here, as usual, \(m\) denotes Lebesgue measure, and \(\log \zeta(it)\) is defined by continuous variation along lines parallel to the real axis, using (1.11.9) for \(\sigma > 1\). By contrast, the corresponding result for \(\sigma = \frac{1}{2}\) states that
\[
\frac{1}{2\pi} \cdot m \{ t \in [-T, T] : \log \zeta(\frac{1}{2} + it) \in R \} \cdot \frac{1}{\sqrt{h[\log(2 + |t|)]}} \cdot \int_{-\infty}^{\infty} e^{-2\pi r^2 / \log T} \, dr \quad (T \to \infty).
\]

(The right-hand side gives a 2-dimensional distribution with mean 0 and variance 1.) This is an unpublished theorem of Selberg, which may be obtained via the method of Ghosh [3].
XII

DIVISOR PROBLEMS

12.1. The divisor problem of Dirichlet is that of determining the asymptotic behaviour as \( x \to \infty \) of the sum

\[
D(x) = \sum_{n \leq x} d(n),
\]

where \( d(n) \) denotes, as usual, the number of divisors of \( n \). Dirichlet proved in an elementary way that

\[
D(x) = x \log x + (2\gamma - 1)x + O(x^{1/2}).
\]

In fact

\[
D(x) = \sum_{n \leq x} 1 - \sum_{n \leq x} \sum_{1 < a < n, \ n = \langle x \rangle} 1
\]

\[
= [\langle x \rangle] + 2 \sum_{n \leq x} \left( [\langle x \rangle] - [\langle x \rangle] \right)
\]

\[
= 2 \sum_{n \leq x} \left( [\langle x \rangle] - [\langle x \rangle] \right)
\]

\[
= 2 \sum_{n \leq x} \left( \frac{x}{n} + O(1) \right) - (\langle x \rangle + O(1))
\]

\[
= 2x(\log x + \gamma + O(x^{-1})) + O(x(x + O(\sqrt{x}))),
\]

and (12.1.1) follows. Writing

\[
D(x) = x \log x + (2\gamma - 1)x + \Delta(x)
\]

we thus have

\[
\Delta(x) = O(x^{1/2}).
\]

Later developments have improved this result, but the exact order of \( \Delta(x) \) is still undetermined.

The problem is closely related to that of the Riemann zeta-function.

By (12.1.1) with \( a_n = d(n) \), \( s = 0 \), \( T \to \infty \), we have

\[
D(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s) x^s}{s} \, ds (c > 1),
\]

provided that \( x \) is not an integer. On moving the line of integration to the left, we encounter a double pole at \( s = 1 \), the residue being

\[
\Delta(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s) x^s}{s} \, ds (0 < c < 1).
\]

12.1. Divisor Problems

The more general problem of

\[
D_k(x) = \sum_{n \leq x} d_k(n),
\]

where \( d_k(n) \) is the number of ways of expressing \( n \) as a product of \( k \) factors, was also considered by Dirichlet. We have

\[
D_k(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(w) x^w \, dw \quad (c > 1).
\]

Here there is a pole of order \( k \) at \( w = 1 \), and the residue is of the form \( x^{k-1} \log x \), where \( x^k \) is a polynomial of degree \( k-1 \). We write

\[
D_k(x) = x^{k-1} \log x + \Delta_k(x),
\]

so that \( \Delta_k(x) = \Delta_k(x) \).

The classical elementary theorem of the subject is

\[
\Delta_k(x) = O(x^{k-1/2} \log^{1/2} x) \quad (k = 2, 3, \ldots).
\]

We have already proved this in the case \( k = 2 \). Now suppose that it is true in the case \( k-1 \). We have

\[
D_k(x) = \sum_{a \leq \sqrt{x}} \sum_{n \leq x/a} \frac{d_{k-1}(n)}{n} + \sum_{n \leq x} d_{k-1}(n)
\]

\[
= \sum_{n \leq x} \sum_{a | n} d_{k-1}(n/a) \frac{a^{k-1}}{a} \sum_{n \leq x} \frac{d_{k-1}(n/a)}{n} \sum_{d_{k-1}(n/a) \leq a} 1
\]

\[
= \sum_{n \leq x} \sum_{a | n} \frac{d_{k-1}(n/a)}{n} \left( \frac{x}{n} + O(1) \right) d_{k-1}(n/a)
\]

\[
= \sum_{n \leq x} \sum_{a | n} \frac{d_{k-1}(n/a)}{n} - x^{k-1} D_{k-1}(x^{k-1}) + O(D_{k-1}(x^{k-1})).
\]

Let us denote by \( p_k(z) \) a polynomial in \( z \), of degree \( k-1 \) at most, not always the same one. Then

\[
\sum_{n \leq x} \frac{\log x \times n}{m} = p_k(\log x) + O(x^{k-1} \log^{1/2} x).
\]

Hence

\[
\sum_{n \leq x} \frac{\log x \times n}{m} = x \log x + O(x^{k-1} \log^{1/2} x).
\]

Also

\[
\sum_{n \leq x} \Delta_{k-1}(n) = O(x^{k-1} \log^{1/2} x \sum_{n \leq x} \frac{1}{n^{1-k/2}})
\]

\[
= O(x^{k-1} \log^{1/2} x \sum_{n \leq x} \frac{1}{n^{1-k/2}})
\]

\[
= O(x^{k-1} \log^{1/2} x).
\]

\[\dagger\] See e.g. Landau (9).
The next term is
\[ x \sum_{n \leq x} \frac{\Delta_2(n) - \Delta_2(n-1)}{n} \approx z \sum_{n \leq x} \frac{\Delta_2(n)}{n(n+1)} + \frac{\Delta_2(x)}{x} \approx \frac{\Delta_2(x)}{N} + \frac{1}{N+1}, \]
where \( N = \lfloor x-1 \rfloor \). Now
\[ x \sum_{n \leq x} \frac{\Delta_2(n) \log n}{n} + \frac{x^2 \Delta_2(x)}{N+1} \approx \sum_{n \leq x} \frac{\Delta_2(n)}{n+1} \log n + \frac{\Delta_2(x)}{x} \log x \]
and
\[ x \sum_{n \leq x} \frac{\Delta_2(n)}{n+1} \log n \approx \sum_{n \leq x} \frac{\Delta_2(n)}{n+1} \log n + \frac{\Delta_2(x)}{x} \log x \]

Finally
\[ x^k \sum_{n \leq x} \Delta_2(n) = x^k \sum_{n \leq x} \Delta_2(n) - x^k \sum_{n \leq x} \Delta_2(n-1) \log (\lfloor x \rfloor - 1), \]

This proves (12.1.4).

We define the order \( a_k \) of \( \Delta_2(n) \) as the least number such that
\[ \Delta_2(n) \approx (\log n)^{a_k}. \]
for every positive \( s \). Thus it follows from (12.1.4) that
\[ a_k \geq \frac{k-1}{k} \quad (k = 2, 3, \ldots). \]

The exact value of \( a_k \) has not been determined for any value of \( k \).

12.2. The simplest theorem which goes beyond this elementary result is

**Theorem 12.2.**

\[ a_k \leq \frac{k-1}{k+1} \quad (k = 2, 3, \ldots). \]

Take \( a_k = a_k(n) = n \), \( a_k = k \), \( x = 0 \), and let \( z > \frac{1}{2} \) be an odd integer, in Lemma 2.12. Replacing \( x \) by \( z \), this gives
\[ D_k(x) = \frac{1}{2\pi i} \int_{c-iT}^{
uline{+iT}} \frac{\zeta'(s)}{\zeta(s-1)} \frac{x^s}{s} \, ds + O \left( \frac{x^{1+\epsilon}}{\log x} \right) + O \left( \frac{x^{1+\epsilon}}{x^2} \right) \quad (\epsilon > 0). \]

Now take the integral round the rectangle \( -a-iT, c-iT, c+iT, -a+iT, \) \( -a+iT \), where \( a > 0 \). We have, by (6.1.1) and the Phragmèn-Lindelöf principle,
\[ \zeta(s) = O \left( (\log x)^{-\delta} \right) \]
in the rectangle. Hence
\[ \int_{-a-iT}^{c+iT} \frac{\zeta(s)}{s} \, ds = O \left( \int_0^T \left( \log x \right)^{1/2-\epsilon} + O \left( T \log T \right) \right) \]

since the integrand is a maximum at one end or the other of the range of integration. A similar result holds for the integral over \(-a \to c\).

The residue at \( x = 1 \) is \( \frac{1}{2}\eta(0) \), and the residue at \( x = 0 \) is \( \eta(0) = O(1) \).

Finally
\[ \int_{-a-iT}^{c+iT} \frac{\zeta(s)}{s} \, ds = \int_{-a-iT}^{c+iT} \frac{\chi(s)}{s} \, ds = \int_{-a+iT}^{c+iT} \frac{\chi(s)}{s} \, ds = \int_{-a-iT}^{c+iT} \frac{\chi(s)}{s} \, ds \]

For \( 1 \leq t \leq T \),
\[ x(-a + it) = C \zeta(\frac{1}{2} + it - 1 + it - 1) + O \left( T \log T \right) \]
and
\[ \frac{1}{-\frac{1}{2} + it} + O \left( \frac{T}{\log T} \right) \]

The corresponding part of the integral is therefore
\[ \int_{-a-iT}^{c+iT} \frac{\chi(s)}{s} \, ds = \int_{-a-iT}^{c+iT} \frac{\chi(s)}{s} \, ds + O \left( T \log T \right), \]

provided that \( (a + 1)k > 1 \). This integral is of the form considered in Lemma 4.5, with
\[ F(t) = \frac{1}{2\pi i} \log \left( \frac{\log (2\pi + 1) + 1 + \log n}{\log n} \right) \]

Since
\[ F(t) = \frac{k}{2} \leq 1 \]
the integral is
\[ O \left( T \log T \right) \].
uniformly with respect to $a$ and $x$. A similar result holds for the integral over $\{ T^s, 1 \}$, while the integral over $\{ 1, T \}$ is bounded. Hence

$$
\Delta(x) = O\left( \frac{x^2}{T^{\frac{1}{4}}(x-1)^2} \right) + O\left( \frac{x^{a+1}}{T} \right) + O\left( \frac{T^{\theta+\frac{1}{2}(a+1)}}{x^2} \right) + \sum_{\nu=1}^{\infty} d_{\nu}(x)O(T^{\nu+\frac{1}{2}(a+1)}).
$$

Taking $\epsilon = 1 - \epsilon$, $a = e$, the terms are of the same order, apart from $\epsilon$'s, if

$$T = x^{3a+1}.
$$

Hence

$$\Delta(x) = O(x^{3a+1}).
$$

The restriction that $x$ should be half an odd integer is clearly unnecessary to the result.

12.3. By using some of the deeper results on $\zeta(x)$ we can obtain a still better result for $k \geq 4$.

**Theorem 12.3.†**

$$a_k \leq \frac{k-1}{k+2} \quad (k = 4, 5, \ldots).
$$

We start as in the previous theorem, but now take the rectangle as far as $\epsilon = \frac{1}{2}$ only. Let us suppose that

$$\zeta(1+it) = O(1),
$$

uniformly in the rectangle. The horizontal sides therefore give

$$O\left( \int \frac{T^{(3a+1)x} - x^{a} \, dt}{T^{2x}} \right) = O(T^{(3a+1)x}x^{a} + O(T^{-2x}).
$$

Also

$$O\left( \int \frac{\psi(x)}{x^{a}} \, dt \right) = O(\chi(x)\sqrt{T}).
$$

Now

$$\int \frac{\psi(x)}{x^{a}} \, dt \leq \max_{\nu=1,2} \left( \frac{T^{1/4} + x^3}{x^{1/2}} \right) \int \frac{\psi(x)}{x^{a}} \, dt = O\left( \frac{T^{1/4} + x^3}{x^{1/2}} \right).$$

† Hardy and Littlewood (4).
where
\[ F(t) = 2 \log(1 + \log(2n + 1) + \log(nz)), \]
\[ F'(t) = \log(1 + \log(2n + 1)). \]

Hence \( F'(t) \geq \log \frac{n}{N + 1}, \) and (12.4.3) is
\[
\frac{1}{\log(n/(N + 1))} \left( \frac{2n}{\log(n/(N + 1))} \right) + O(\gamma n) = O\left( \frac{n^2}{\log(n/(N + 1))} \right).
\]

For \( n > 2N \) this contributes to (12.4.4)
\[
\frac{2n^2}{\log(n/(N + 1))} = O(n^2).
\]

For \( n > n > 2N \) it contributes
\[
O\left( \frac{n^2}{\log(n/(N + 1))} \right) = O(n^2).
\]

Similarly for the integral over \(-T < t \leq -1; \) and the integral over \(-1 < t \leq 1 \) is clearly \( O(n^2). \)

If \( n < N, \) we write
\[
\begin{aligned}
&\frac{\cos(4\pi x/n) - \cos(4\pi x/n)}{n} \\
&\quad = \frac{\cos(4\pi x/n) - \cos(4\pi x/n)}{n} + \frac{\cos(4\pi x/n) - \cos(4\pi x/n)}{n}.
\end{aligned}
\]

The first term is
\[
\frac{1}{n} \int_{-T}^{T} \cos(4\pi x/n) \Gamma(n/(N + 1)) - \Gamma(n/(N + 1)) \cos(4\pi x/n) \, dx
\]
\[
= -\frac{1}{n^2} \int_{-T}^{T} \cos(4\pi x/n) \Gamma(n/(N + 1)) - \Gamma(n/(N + 1)) \cos(4\pi x/n) \, dx
\]
\[
= -4i \int_{0}^{T} \left( K_1(4\pi x/n) \right) - \Gamma(n/(N + 1)) \, dx.
\]

in the usual notation of Bessel functions.†

The first integral in the brackets is
\[
\frac{1}{\pi} \int_{0}^{T} \left( A + i + O(1) \right) \, dx = O\left( \frac{1}{\log(n/N + 1)} \right),
\]
which gives
\[
\sum_{x=1}^{n} \frac{d(n)}{\log(n/N + 1)} = O(n^2).
\]

† See, e.g., Titchmarsh, *Fourier Integrals*, (7.3.11), (7.9.11).


Hence, multiplying by \( e^{it} \) and taking the real part,

\[
\sum_{n \neq 0} d(n) \cos(4\pi z(nz - 1)) - O(N^{1/2} + \delta) = O(N^{1/2} + \delta).
\]

Using this and partial summation, (12.4.4) gives

\[
\Delta(\mathbf{c}) = O(\mathbf{N^{1/2}} + \delta) + O(N^{1/2} + \delta) + O(N\mathbf{N^{1/2}}) + O(N^{1/2} + \delta) = O(\mathbf{N^{1/2}} + \delta) + O(N\mathbf{N^{1/2}}) + O(N^{1/2} + \delta).
\]

Taking \( \sigma = 1, \epsilon = 1 - \varepsilon, \) the first and last terms are of the same order, apart from \( \delta \) and if

\[
N = [\mathbf{\frac{1}{[2\pi]}^\sigma}]^2,
\]

Hence

\[
\Delta(\mathbf{c}) = O(\mathbf{N^{1/2}} + \delta),
\]

the result stated.

A similar argument may be applied to \( \Delta_{\mathbf{a}}(\mathbf{c}) \). We obtain

\[
\Delta_{\mathbf{a}}(\mathbf{c}) = \sum_{\mathbf{n} < N_{\mathbf{a}}} \mathbf{d}(\mathbf{n}) \cos(\mathbf{a}(nz - 1)) + O(\mathbf{N^{1/2}} + \delta),
\]

and deduce

\[\mathbf{a} \leq \mathbf{N}^{1/2}\]

The detailed argument is given by Atkinson (3).

If the series in (12.4.4) were absolutely convergent, or if the terms were so loosely cancelled that \( \mathbf{a} \leq \mathbf{1} \) and it may reasonably be conjectured that this is the real truth. We shall see later that \( \mathbf{a} \leq \mathbf{1} \), so that it would follow that \( \mathbf{a} \leq \mathbf{1} \). Similarly from (12.4.6) we should obtain \( \mathbf{a} \leq \mathbf{1} \); and so generally it may be conjectured that

\[
\mathbf{a} = \mathbf{\frac{k-1}{2k}}
\]

12.5. The average order of \( \Delta_{\mathbf{a}}(\mathbf{c}) \). We may define \( \beta_{\mathbf{a}} \), the average order of \( \Delta_{\mathbf{a}}(\mathbf{c}) \), to be the least number such that

\[
\mathbf{1} \frac{1}{2} \int \Delta_{\mathbf{a}}(\mathbf{c}) \mathbf{d}y = O(\mathbf{y^{\beta_{\mathbf{a}}+\epsilon}})
\]

for every positive \( \epsilon \). Since

\[
\mathbf{1} \frac{1}{2} \int \Delta_{\mathbf{a}}(\mathbf{c}) \mathbf{d}y = \mathbf{1} \frac{1}{2} \int \mathbf{O}(\mathbf{c^{2\beta_{\mathbf{a}}+\epsilon}}) \mathbf{d}y = O(\mathbf{c^{2\beta_{\mathbf{a}}+\epsilon}}),
\]

we have \( \beta_{\mathbf{a}} \leq \mathbf{a} \) for each \( \mathbf{b} \). In particular we obtain a set of upper bounds for the \( \beta_{\mathbf{a}} \) from the above theorems.

As usual, the problem of average order is easier than that of order, and we can prove more about the \( \beta_{\mathbf{a}} \) than about the \( \mathbf{a} \). We shall first prove the following theorem:

\[\beta_{\mathbf{a}} \leq \mathbf{a} \leq \mathbf{2}\]

\[\beta_{\mathbf{a}} \leq \mathbf{a} \leq \mathbf{2}\]

12.5. Theorem: Let \( \gamma_{\mathbf{b}} \) be the lower bound of positive numbers \( \mathbf{a} \) for which

\[
\frac{1}{2\pi} \int_{|z| = \mathbf{1}} \left| \frac{\mathbf{e}^{\mathbf{z}}}{\mathbf{z}} \right|^{\mathbf{1/2}} \mathbf{d}z < \infty.
\]

Then \( \beta_{\mathbf{a}} = \mathbf{2} \), and

\[
\frac{1}{2\pi} \int_{|z| = \mathbf{1}} \left| \frac{\mathbf{e}^{\mathbf{z}}}{\mathbf{z}} \right|^{\mathbf{1/2}} \mathbf{d}z = \int_{0}^{\infty} \Delta_{\mathbf{b}}(\mathbf{c})y^{\beta_{\mathbf{a}}-1} \mathbf{d}y,
\]

provided that \( \mathbf{a} > \mathbf{1} \).

We have

\[
\Delta_{\mathbf{b}}(\mathbf{c}) = \frac{1}{2\pi} \lim_{\mathbf{N} \to \infty} \int_{0}^{\mathbf{N}} \mathbf{C}\left( \frac{\mathbf{N}}{\mathbf{a}} \right) \mathbf{d}y.
\]

Applying Cauchy’s theorem to the rectangle \( \gamma - \mathbf{1}, \gamma + \mathbf{1}, \gamma + \mathbf{iT}, \gamma - \mathbf{iT}, y - \mathbf{iT}, y + \mathbf{iT}, y + \mathbf{iT}, y - \mathbf{iT}, \) where \( y \) is less than, but sufficiently near to, \( \mathbf{1} \), and allowing for the residue at \( \mathbf{z} = \mathbf{1} \), we obtain

\[
\Delta_{\mathbf{b}}(\mathbf{c}) = \frac{1}{2\pi} \lim_{\mathbf{N} \to \infty} \int_{\gamma - \mathbf{iT}}^{\gamma + \mathbf{iT}} \mathbf{C}(\mathbf{y}) \mathbf{d}y.
\]

Actually (12.5.3) holds for \( \gamma_{\mathbf{b}} < \gamma < \mathbf{1} \). For \( \mathbf{y}(\mathbf{z}) \to 0 \) uniformly as \( y \to \pm\infty \) in the strip. Hence we integrate the integrand of (12.5.3) round the rectangle \( \gamma - \mathbf{iT}, \gamma + \mathbf{iT}, \gamma - \mathbf{iT}, \gamma + \mathbf{iT}, \) where

\[
\gamma_{\mathbf{b}} < \gamma < \mathbf{1},
\]

and make \( T \to \infty \), we obtain the same results with \( \gamma_{\mathbf{b}} \) instead of \( \gamma \).

If we replace \( \mathbf{a} \) by \( \mathbf{1} \), (12.5.3) expresses the relation between the Mellin transform

\[
\mathbf{f}(\mathbf{z}) = \mathbf{\Delta}_{\mathbf{b}}(\mathbf{1}) \mathbf{g}(\mathbf{z}),
\]

the relevant integrals holding also in the mean-square sense. Hence Parseval’s formula for Mellin transform gives

\[
\mathbf{1} \frac{1}{2\pi} \int_{|z| = \mathbf{1}} \left| \frac{\mathbf{e}^{\mathbf{z}}}{\mathbf{z}} \right|^{\mathbf{1/2}} \mathbf{d}z = \int \mathbf{\Delta}_{\mathbf{b}}(\mathbf{1}) \mathbf{y}^{\beta_{\mathbf{a}}-1} \mathbf{d}y = \int \mathbf{\Delta}_{\mathbf{b}}(\mathbf{a}) \mathbf{y}^{\beta_{\mathbf{a}}-1} \mathbf{d}y = \int \mathbf{y}^{\beta_{\mathbf{a}}-1} \mathbf{d}y
\]

provided that \( \gamma_{\mathbf{b}} < \gamma < \mathbf{1} \).

It follows that, if \( \gamma_{\mathbf{b}} < \gamma < \mathbf{1} \),

\[
\int \mathbf{\Delta}_{\mathbf{b}}(\mathbf{a}) \mathbf{y}^{\beta_{\mathbf{a}}-1} \mathbf{d}y < K(\mathbf{K} \mathbf{y}^{\beta_{\mathbf{a}}-1}),
\]

1 By an application of the lemma of § 11.2.

2 See Titchmarsh, Theory of Fourier Integrals, Theorem 71.
and, replacing $X$ by $\frac{1}{2}X$, $\frac{1}{4}X$, ..., and adding,

$$\frac{\Delta_k(x)}{x} = \int \Delta_k(x) dx = KX^{\nu-1}.\]

Hence $\beta_2 \leq \gamma$, and so $\beta_2 \leq \gamma_2$.

The inverse Mellin formula is

$$\oint \Delta_k(x) e^{-x} dx = \int \Delta_k(x) e^{-x} dx.\]

The right-hand side exists primarily in the mean-square sense, for $\gamma_2 < \sigma < 1$. But actually the right-hand side is uniformly convergent in any region interior to the strip $\beta_2 < \sigma < 1$; for

$$\int \Delta_k(x) e^{-x} dx = \int \Delta_k(x) e^{-x} dx\]

$$(O(X^{\nu+\epsilon} + O(X^{-\nu-1}))) = O(X^{\nu+\epsilon}).$$

and on putting $X = 2, 4, 8, ...$, and adding we obtain

$$\int \Delta_k(x) e^{-x} dx \leq \int \Delta_k(x) e^{-x} dx < K.$$

It follows that the right-hand side of (12.5.5) represents an analytic function, regular for $\beta_2 < \sigma < 1$. The formula therefore holds by analytic continuation throughout this strip. Also (by the argument just given) the right-hand side of (12.5.4) is finite for $\beta_2 < \gamma < 1$. Hence so is the left-hand side, and the formula holds. Hence $\gamma_2 \leq \beta_2$, and so, in fact, $\gamma_2 = \beta_2$. This proves the theorem.

**12.6. Theorem 12.6 (A).**

If $\beta_2 \geq \frac{k-1}{2k}$, $k = 2, 3, ...$.

If $\frac{1}{4} < \sigma < 1$, by Theorem 7.2,

$$\zeta(T) \geq \int \frac{1}{\Gamma} \left| \frac{\Delta_k(x + i) \Delta_k(x - i)}{x^2} \right| \left| \frac{\Delta_k(x) e^{-x}}{x^2} \right|^{1/2}.$$

Hence

$$\int \frac{1}{\Gamma} \left| \frac{\Delta_k(x + i) \Delta_k(x - i)}{x^2} \right| \left| \frac{\Delta_k(x) e^{-x}}{x^2} \right|^{1/2} \leq \zeta(T).$$

**12.6. Theorem 12.6 (B).**

Hence, if $0 < \sigma < 1$, $T > 1$,

$$\int \left| \frac{\Delta_k(x + i) \Delta_k(x - i)}{x^2} \right| \left| \frac{\Delta_k(x) e^{-x}}{x^2} \right|^{1/2} \leq \int \left| \frac{\Delta_k(x + i) \Delta_k(x - i)}{x^2} \right| \left| \frac{\Delta_k(x) e^{-x}}{x^2} \right|^{1/2} \leq \zeta(T).$$

This can be made as large as we please by choice of $T$ if $\sigma < \frac{1}{2}(1/k)$. Hence

$$\gamma_2 \geq \frac{k-1}{2k},$$

and the theorem follows.

**Theorem 12.6 (C).**

For $\sigma_0 \geq \beta_2$.

Much more precise theorems of the same type are known. Hardy proved first that both

$$\Delta(x) > K\zeta, \quad \Delta(x) < -K\zeta$$

hold for some arbitrarily large values of $x$, and then that $\zeta$ may in each case be replaced by

$$\sigma \log x \log \log x.$$

12.7. We recall that (7.09) the numbers $\delta_0$ are defined as the lower bounds of $\sigma$ such that

$$\zeta(T) \geq \frac{1}{\Gamma} \int \frac{1}{\Gamma} \left| \frac{\Delta_k(x + i) \Delta_k(x - i)}{x^2} \right| \left| \frac{\Delta_k(x) e^{-x}}{x^2} \right|^{1/2} = O(1).$$

We shall now prove

**Theorem 12.7.**

For each integer $k \geq 2$, a necessary and sufficient condition that

$$\beta_2 \leq \frac{k-1}{2k + 1}$$

is that

$$\sigma_0 \leq \frac{k+1}{2k + 1}.$$
for \( \sigma < \frac{k(k-1)}{k} \). It follows from the convexity of mean values that

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^{\frac{k(k-1)}{k}+\min(\sigma-\frac{k(k-1)}{k}, \frac{1}{2k})})
\]

for

\[
\frac{k-1-\varepsilon}{2k} < \sigma < \frac{k+1+\varepsilon}{2k}.
\]

The index of \( T \) is less than 2 if

\[
\sigma > \frac{k-1+\varepsilon}{2k}.
\]

Then

\[
\int_{T} \frac{|(\sigma+i\delta)^k|}{|\sigma+i\delta|^2} \, dt = O(T^{-\delta}) \quad (\delta > 0).
\]

Hence (12.5.1) holds. Hence \( \gamma_k < \frac{k(k-1)}{k} \). Hence \( \beta_k < \frac{k(k-1)}{2k} \), and so, by Theorem 12.6(a), (12.7.1) holds.

On the other hand, if (12.7.1) holds, it follows from (12.5.2) that

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^\rho)
\]

for \( \sigma > \frac{k(k-1)}{k} \). Hence by the functional equation

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^{\frac{k(k-1)}{k}+\delta})
\]

for \( \sigma < \frac{k(k+1)}{k} \). Hence, by the convexity theorem, the left-hand side is \( O(T^{\rho+\delta}) \) for \( \sigma = \frac{k(k+1)}{k} \); hence, in the notation of § 7.9, \( \alpha_k \leq \frac{k(k+1)}{k} \), and so (12.7.2) holds.


\( \beta_k = \frac{1}{2}, \quad \beta_k = \frac{1}{2}, \quad \beta_k = \frac{1}{2} \).

By Theorem 7.7, \( \gamma_k < \frac{k(k-1)}{k} \). Since

\[
\frac{1}{2} < \frac{k+1}{2k} \quad (k < 3)
\]

it follows that \( \beta_k = \frac{1}{2}, \beta_k = \frac{1}{2}, \beta_k = \frac{1}{2} \).

The available material is not quite sufficient to determine \( \beta_k \). Theorem 12.5(A) gives \( \beta_k = \frac{1}{2} \). To obtain an upper bound for it, we observe that, by Theorem 6.6 and (7.6.1),

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^{\rho+\delta}) \quad \text{for} \quad \gamma_k < \frac{k(k-1)}{k}.
\]

\( \dagger \) The value of \( \beta_k \) is due to Hardy (3), and that of \( \beta_k \) to Cramér (4); for \( \beta_k \), see Titchmarsh (22).

and, since \( \gamma_k < \frac{k}{2} \) by Theorem 7.10,

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^{\rho}) \quad \text{for} \quad \gamma_k < \frac{k}{2}.
\]

Hence by the convexity theorem

\[
\int_{T} |(\sigma+i\delta)^k| \, dt = O(T^{\rho+\delta})
\]

for \( \delta = \frac{1}{k} < \frac{1}{2} \). It easily follows that \( \gamma_k < \frac{1}{2}, \) i.e. \( \beta_k < \frac{1}{2} \).

NOTES FOR CHAPTER 12

12.9. For large \( k \) the best available estimates for \( \gamma_k \) are of the shape \( \gamma_k \approx 1 - C \chi^{-k} \), where \( \chi \) is a positive constant. The first such result is due to Richert (2). (See also Karamata (1), Ivić (3, Theorem 13.3) and Fujii (31).) These results depend on bounds of the form (6.19.3).

For the range \( 4 < k < 8 \) one has \( \gamma_k \approx \frac{1}{2} - 1/k \) (Heath-Brown [8]) while for intermediate values of \( k \) a number of estimates are possible (see Ivić (3, Theorem 13.2)). In particular one has \( \gamma_k \approx \frac{k}{2}, \gamma_k \approx \frac{k}{2}, \gamma_k \approx \frac{k}{2} \), and \( \gamma_k \approx \frac{k}{2} \).

12.10. The following bounds for \( \gamma_k \) have been obtained.

\[
\begin{align*}
\dagger & = 0.330000 \ldots \quad \text{van der Corput (9),} \\
\dagger & = 0.320000 \ldots \quad \text{van der Corput (4),} \\
\dagger & = 0.320000 \ldots \quad \text{Chih (7),} \quad \text{Kolesnik (8).} \\
\dagger & = 0.324220 \ldots \quad \text{Kolesnik (1),} \\
\dagger & = 0.324273 \ldots \quad \text{Kolesnik (2),} \\
\dagger & = 0.324074 \ldots \quad \text{Kolesnik (4),} \\
\dagger & = 0.324000 \ldots \quad \text{Kolesnik (8).}
\end{align*}
\]

In general the methods used to estimate \( \gamma_k \) and \( \mu(x) \) are very closely related. Suppose one has a bound

\[
\sum_{N < n < x} \sum_{\chi(n)} \exp[2\pi i (x(n) - x^{-1}(mn)^{1/2})] = (MN)^{1/2} + \Omega(MN^{1/2})
\]

for any constant \( c \), uniformly for \( M < N \), \( M < N \), \( N \leq N \), and \( MN \approx x^{1/2} \). It then follows that \( \gamma_k \approx \frac{1}{2}, \gamma_k \approx \frac{1}{2}, \gamma_k \approx \frac{1}{2} \), and \( \mu(x) \approx \frac{1}{2} \) (for \( E(T) \) as in § 7.29). In practice those versions of the van der Corput
method used to tackle \( \mu(n) \) and \( \sigma_1 \) also apply to (12.10.1), which explains the similarity between the table of estimates given above and that presented in §5.21 for \( \sigma(n) \). This is just one manifestation of the close similarity exhibited by the functions \( \Omega(T) \) and \( \Delta(x) \), which has its origin in the formulae (7.20.6) and (12.4.4). The classical lattice-point problem for the circle falls within the same area of ideas. Thus, if the bound (12.10.3) holds, along with its analogue in which the summation condition \( m = 1 \) (mod 6) is imposed, then one has

\[ g \equiv (m, n) \equiv 2; \quad m^2 + n^2 < x \Rightarrow xx + O(x^{2/3}). \]

Jutila [2] has taken these ideas further by demonstrating a direct connection between the size of \( \Delta(x) \) and that of \( \zeta(1 + it) \) and \( E(T) \leq T^{1/2 - \varepsilon} \). Further work has been done on the problems of estimating \( \sigma_1 \). The best results at present is \( x < \frac{1}{2} \), due to Kolesnik [3]. For \( x \), however, no sharpening of the bound \( x < \frac{1}{2} \) given by Theorem 12.3 has yet been found. This result, dating from 1952, seems very resistant to any attempt at improvement.

12.11. The \( \Omega \)-results attributed to Hardy in §12.6 may be found in Hardy [1]. However Hardy’s arguments appear to yield only

\[ \Delta(x) = \Omega_{\beta}(t \log x) \left( \log \log x \right)^{1-\beta}, \]  

(12.11.1)

and not the corresponding \( \Omega \) result. The reason for this is that Dirichlet’s theorem is applicable for \( \beta \), while Zeeck’s theorem is needed for the \( \Omega \) result. By using a quantitative form of Kronecker’s theorem, Coradi and Kápari [1] showed that

\[ \Delta(x) = \Omega_{\beta}(t \log x) \left( \log \log x \right)^{1-\beta}, \]

for a certain positive constant \( c \). This improved earlier work of Ingham [1] and Gangadhara [1]. Hardy’s result (12.11.1) has also been sharpened by Hafner [1] who obtained

\[ \Delta(x) = \Omega_{\beta}(t \log x) \left( \log \log x \right)^{1/2 + \varepsilon} \exp \left( -c \log \log \log x \right)^2 \]

for a certain positive constant \( c \). For \( x \geq 3 \) he also showed [2] that, for a suitable positive constant \( c \), one has

\[ \Delta(x) = \Omega_{\beta}(t \log x)^{1/2 + \varepsilon} \left( \log \log x \right)^{1/2} \exp \left( -c \log \log \log x \right)^2 \].

12.12. As mentioned in §7.28, we now have \( \sigma_1 < \frac{1}{2} \), whereas \( \beta_1 = \frac{1}{2} \). Heath-Brown [3] has shown that \( \beta_1 = \frac{1}{2} \). For \( k = 2 \) and 3 one can give asymptotic formulae for

\[ \int_0^x \Delta(x) dy. \]

Thus Tong [1] showed that

\[ \int_0^x \Delta(x) dy \sim \frac{x^{\frac{3k-1}{2k}}}{(2k-2)x^k} \sum_{n=1}^x \frac{d_n(n)^{\frac{1}{2} - k}}{n^{k-1}} + R_k(x) \]

with \( R_k(x) \sim x \log x \) and

\[ R_k(x) \sim x \log x \]

(12.11.1)

and \( \Omega_{\beta} \) is \( \Omega_{\beta} \) for \( k = 3 \) and \( \Omega_{\beta} \) for \( k > 4 \).

Taking \( c_1 \leq \frac{1}{2} \) (see §7.20) yields \( c_1 < \frac{1}{2} \). However the available information concerning \( \sigma_1 \) is as yet insufficient to give \( c_1 < \frac{3k-1}{2k} \) for any \( k > 4 \). It is perhaps of interest to note that Hardy’s result (12.11.1) implies \( R_k(x) \sim x \log \log x \), since any estimate \( R_k(x) \sim x \log x \) easily leads to a bound \( \Delta(x) \sim \left( F(x) \log x \right)^{1/2} \), by an argument analogous to that given for the proof of Lemma 2 in §14.13.

Ivic [3, Theorem 13.9 and 13.10] has estimated the higher moments of \( \Delta(x) \) and \( \Delta_2(x) \). In particular his results imply that

\[ \int_0^x \Delta(x) dy < x^{1/2}. \]

For \( \Delta(x) \) his argument may be modified slightly to yield

\[ \int_0^x |\Delta(x)|^{1/2} dy < x^{1/2}. \]

Those results are readily seen to contain the estimates \( \sigma_2 \leq \frac{1}{2} \), \( \beta_1 \leq \frac{1}{2} \) and \( \sigma_2 \leq \frac{1}{2} \), \( \beta_1 \leq \frac{1}{2} \) respectively.
XIII

THE LINDELÖF HYPOTHESIS

13.1. The Lindelöf hypothesis is that

$$|x + iy| = O(r^n)$$

for every positive $r$; or, what comes to the same thing, that

$$|x + iy| = O(r^\sigma)$$

for every positive $r$ and every $\sigma \geq \frac{1}{2}$; for either statement is, by the theory of the function $\mu(z)$, equivalent to the statement that $\mu(0) = 0$ for $\sigma \geq \frac{1}{2}$. The hypothesis is suggested by various theorems in Chapters V and VII. It is also the simplest possible hypothesis on $\mu(z)$, for on it the graph of $y = \mu(x)$ consists simply of the two straight lines

$$y = 1 - \sigma \quad (\sigma \leq \frac{1}{2}), \quad y = 0 \quad (\sigma \geq \frac{1}{2}).$$

We shall see later that the Lindelöf hypothesis is true if the Riemann hypothesis is true. The converse deduction, however, cannot be made — in fact (Theorem 13.6) the Lindelöf hypothesis is equivalent to a much less drastic, but still unproved, hypothesis about the distribution of the zeros.

In this chapter we investigate the consequences of the Lindelöf hypothesis. Most of our arguments are reversible, so that we obtain necessary and sufficient conditions for the truth of the hypothesis.

13.2. Theorem 13.2.1. Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are

$$\int_1^\infty |x + it|^{2k} dt = O(T^n) \quad (k = 1, 2, \ldots); \quad (13.2.1)$$

$$\int_1^\infty |x + it|^{2k} dt = O(T^{\sigma}) \quad (\sigma \geq \frac{1}{2}, \quad k = 1, 2, \ldots); \quad (13.2.2)$$

$$\int_1^\infty |x + it|^{2k} dt = \sum_{n=1}^\infty n^\sigma \quad (\sigma \geq \frac{1}{2}, \quad k = 1, 2, \ldots). \quad (13.2.3)$$

The equivalence of the first two conditions follows from the convexity theorem (§ 7.8), while that of the last two follows from the analysis of § 7.9. It is therefore sufficient to consider (13.2.1).

† Hardy and Littlewood (8).

The necessity of the condition is obvious. To prove that it is sufficient, suppose that $|x + it|$ is not $O(r^n)$. Then there is a positive number $\lambda$, and a sequence of numbers $a + ib_k$, such that $t \to \infty$ with $\lambda$, and

$$|a + ib_k| > \lambda^k \quad (C' > 0).$$

On the other hand, on differentiating (2.1.4) we obtain, for $t \geq 1$,

$$|y(x + it)| < Bt,$$

$E$ being a positive absolute constant. Hence

$$|x + it| - |a + ib_k| = \left| \int_a^x (x + it - a + ib_k) \, dt \right| < 2E |x - a| < \frac{1}{2} \lambda^k$$

if $|t - a| < \lambda^{-1}$ and $t$ is sufficiently large. Hence

$$|x + it| > \lambda^k \quad (t - \lambda^{-1} < t < \lambda^{-1}).$$

Take $T = \lambda^{-1}$, so that the interval $(\lambda^{-1} - \lambda^{-2}, \lambda^{-1} + \lambda^{-2})$ is included in $(T, 2T)$ if $t$ is sufficiently large. Then

$$\int_t^{2T} |x + it| dx > \int_{\lambda^{-1}}^{\lambda^{-2}} \lambda^k dx = \frac{1}{2} \lambda^k \lambda^{-1},$$

which is contrary to hypothesis if $\lambda$ is large enough. This proves the theorem.

We could plainly replace the right-hand side of (13.2.1) by $O(T^n)$ without altering the theorem or the proof.

13.3. Theorem 13.2. A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every positive integer $k$ and $\sigma > \frac{1}{2},$

$$\sum_{n=1}^\infty \frac{d(n)}{n^\sigma} + O(t^{-\sigma}) \quad (t > 0), \quad (13.3.1)$$

where $\delta$ is any given positive number less than $1$, and $\lambda - M(k, \delta, \sigma) > 0$.

We may express this roughly by saying that, on the Lindelöf hypothesis, the behaviour of $\zeta(s)$, or of any of its positive integral powers, is dominated, throughout the right-hand half of the critical strip, by a section of the associated Dirichlet series whose length is less than any positive power of $t$, however small. The result may be contested with what we can declare, without unproved hypothesis, from the approximate functional equation.

Taking $d_n = -d(n)$ in Lemma 2.1.2, we have (if $a$ is half an odd integer)

$$\sum_{n=1}^\infty \frac{d(n)}{n^\sigma} = \frac{1}{2 \pi i} \int_{a-iT}^{a+iT} \zeta(s) x^s \frac{ds}{s} + O \left( \frac{x^\rho}{T^{\rho + 1}} \right)$$

where $\rho$ is a positive number less than $\frac{1}{2}$. This is equivalent to the statement for $\zeta(s)$ and $\zeta'(s)$ by subtracting the latter from the former; the higher powers of $x$ are absorbed in the $O$-term.
where \( c > 1 - \sigma + \epsilon \). Now let \( 0 < t < T-1 \), and integrate round the rectangle \( \frac{1}{2} - \sigma - iT, \frac{1}{2} - \sigma + iT, \frac{1}{2} + \sigma + iT, \frac{1}{2} + \sigma - iT \). We have
\[
\frac{1}{2\pi} \int \frac{Q(x+iw) x^m}{w^{s+1}} \, dw = N(s) \int \frac{1}{1-s} \log x \, dx
\]
\[
= L(s) + O(x^{\sigma-\epsilon}(\log x)^{-1}),
\]
\( P \) being a polynomial in its arguments. Also
\[
\int_{-T}^{T} \int_{-T}^{T} |Q(x+iw) x^m| \, dw = O(x^{\sigma T^{-1}})
\]
by the Lindelöf hypothesis; and
\[
\int_{-T}^{T} |Q(x+iw) x^m| \, dw = O\left( x^{-1} \int_{-T}^{T} |Q(x+iw) x^m| \, dw \right)
\]
by the Lindelöf hypothesis. Hence
\[
L(s) = \sum_{n \leq x} \frac{\Lambda(n)}{n^s} \left( \int_{-T}^{T} \frac{x^m}{(x+iw)^s} \right) + O(x^{\sigma T^{-1}}) + O(x^{\sigma T^{-1}})
\]
and (13.3.1) follows on taking \( x = \frac{1}{2} + \frac{1}{2} T \). Conversely, the condition is clearly sufficient, since it gives
\[
L(s) = O\left( \sum_{n \leq x} \frac{1}{n^s} \right) \leq O\left( x^{\sigma T^{-1}} \right) + O(x^{\sigma T^{-1}})
\]
where \( \frac{1}{2} \) is arbitrarily small.

The result may be used to prove the equivalence of the conditions of the previous section, without using the general theorems proved.

13.4. Another set of conditions may be stated in terms of the numbers \( a_k \) and \( b_k \) of the previous chapter.

**Theorem 13.4.** Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are
\[
a_k \leq \frac{1}{2} \quad (k = 2, 3, \ldots), \tag{13.4.1}
\]
\[
b_k < \frac{1}{2} \quad (k = 2, 3, \ldots), \tag{13.4.2}
\]
\[
b_k = \frac{k-1}{2k} \quad (k = 2, 3, \ldots). \tag{13.4.3}
\]

As regards sufficiency, we need only consider (13.4.3), since the other conditions are formally more stringent. Now (13.4.3) gives \( b_k \leq \frac{1}{2} \), and so
\[
\int_{|a+it| = 1} \frac{|Q(a+it)|}{|a+it|^{s+1}} \, ds = O(1) \quad (\sigma > \frac{1}{2}),
\]
\[
\int_{|a+it| = 1} \frac{|Q(a+it)|}{|a+it|^{s+1}} \, ds = O(T^\sigma) \quad (\sigma > \frac{1}{2}).
\]

The truth of the Lindelöf hypothesis follows from this, as in §13.2.

Now suppose that the Lindelöf hypothesis is true. We have, as in §13.2,
\[
D_k(s) = \frac{1}{2\pi i} \int_{-T}^{T} \frac{Q(s,x,x^m)}{x^s} \, dx + O(x^{\sigma T^{-1}}).
\]

Now integrate round the rectangle with vertices at \( \frac{1}{2} + iT, \frac{1}{2} - iT, \frac{1}{2} + iT, \frac{1}{2} - iT \). We have
\[
\int_{-T}^{T} \frac{Q(s,x,x^m)}{x^s} \, dx = O(x^{\sigma T^{-1}}),
\]
\[
\int_{-T}^{T} \frac{Q(x,x^m)}{x^s} \, dx = O\left( \int_{-T}^{T} \frac{|Q(x+iw)|}{|f(x+iw)|} \, dw \right).
\]

The residue at \( s = 1 \) accounts for the difference between \( D_k(s) \) and \( \Delta_k(s) \). Hence
\[
\Delta_k(s) = O(x^{\sigma T^{-1}}) + O(x^{\sigma T^{-1}}).
\]

Taking \( T \to \infty \), it follows that \( b_k \leq \frac{1}{2} \). Hence also \( a_k \leq \frac{1}{2} \). But in fact \( a_k \leq \frac{1}{2} \) on the Lindelöf hypothesis, so that, by Theorem 12.7, (13.4.3) also follows.

13.5. The Lindelöf hypothesis and the zeros.

**Theorem 13.5.** A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every \( \sigma > \frac{1}{2} \),
\[
N(v,T+1) - N(v,T) = O(\log T).
\]

The necessity of the condition is easily proved. We apply Jensen's formula
\[
\log f(z) = \sum_{\gamma \in \gamma} \frac{1}{2\pi i} \int_{|z - \gamma| = r} \frac{f(w)}{w-z} \, dw + \log f(0),
\]
where \( \gamma \) are the moduli of the zeros of \( f(z) \) in \( |z| \leq r \), to the circle with centre \( 2+iT \) and radius \( |z-2+iT| = T \), with \( f(z) \) being \( \zeta(s) \). On the Lindelöf
hypothosis the right-hand side is less than \( o(\log T) \); and, if there are \( N \) zeros in the concentric circle of radius \( \frac{1}{2} \), the left-hand side is greater than
\[ N \log [\log (1-\beta)/\log (1-\delta)]. \]

Hence the number of zeros in the circle of radius \( \frac{1}{2} \) is \( o(\log T) \); and the results stated, with \( \alpha = \frac{1}{2} + \delta \), clearly follows by superposing a number (depending on \( \delta \) only) of such circles.

To prove the converse, let \( G \) be the circle with centre 2+iT and radius \( \frac{1}{2} + \delta \) \( (\beta > 0) \), and let \( \Sigma \) denote a summation over zeros of \( \xi(s) \) in \( G \). Let \( G' \) be the concentric circle of radius \( \frac{1}{2} - \delta \). Then for \( s \) in \( G' \)
\[ \rho(s) = \sum_{\gamma \in \Sigma} \frac{1}{\gamma - s} = O(\log T). \]
This follows from Theorem 9.6 (A), since for each term which is in one of the sums
\[ \sum_{\gamma \in \Sigma} \frac{1}{\gamma - s} = O(\log T), \]
but not in the other, \( |\gamma - s| > \delta \); and the number of such terms is \( O(\log T) \).

Let \( G' \) be the concentric circle of radius \( \frac{1}{2} - \delta \). Then \( \rho(s) = O(\log T) \) for \( s \) in \( G' \), since each term is \( O(1) \), and by hypothesis the number of terms is \( o(\log T) \). Hence Hadamard’s three-circles theorem gives, for \( s \) in \( G' \),
\[ \log |\xi(s)| = o(\log T)^n, \]
where \( a + \beta = 1 \), \( 0 < \beta < 1 \), \( a \) and \( \beta \) depending on \( \delta \) only. Thus in \( G' \),
\[ \rho(s) = o(\log T), \]
for any given \( \delta \).

Now
\[ \int_{1/2}^{1/2+\delta} \rho(s) \, ds = \log \left[ \frac{2+\delta}{2+\delta} \right] - \log \left[ \frac{4+\delta + i\delta}{4+\delta + i\delta} \right] - \int_{1/2}^{1/2+\delta} \rho(s) \, ds = O(1) - \log \left[ \frac{4+\delta + i\delta}{4+\delta + i\delta} \right] + \int_{1/2}^{1/2+\delta} \rho(s) \, ds, \]
since \( \rho(s) \) has \( O(\log T) \) terms. Also, if \( T \to \infty \), the left-hand side is \( o(\log T) \).

Hence, putting \( T = T \) and taking real parts,
\[ \log \left[ \frac{4+\delta + i\delta}{4+\delta + i\delta} \right] = o(\log T) + \int_{1/2}^{1/2+\delta} \rho(s) \, ds, \]
since \( |\gamma = 1 + iT| = A \) in \( G' \), it follows that
\[ \log \left[ \frac{4+\delta + i\delta}{4+\delta + i\delta} \right] = o(\log T), \]
i.e. the Lindelöf hypothesis is true.

\[ \dagger \] Littlewood (4).
work) that the converse of Theorem 13.6(b) follows from Lemma 21 of Selberg (§).

Theorem 13.8. If \( S_\ell(t) = o(\log t) \), then the Lindelöf Hypothesis holds.

We reproduce the arguments used by Selberg and by Ghosh and Goldston here. Let \( \frac{1}{2} < r < 2 \), and consider the integral

\[
\int_0^{t} \log \left( \frac{t + iT}{r^2 + u^2} \right) \frac{du}{u^2}.
\]

Since \( \log \left( \frac{t + iT}{r^2 + u^2} \right) \sim 2 \log t \) the integral is easily seen to vanish, by moving the line of integration to the right. We now move the line of integration to the left, to \( R(s) = s \), passing a pole at \( s = 1 + \frac{1}{2} \), with residue \( -\frac{1}{2} \log (s + iT) = O(1) \). We must make detours around \( s = 1 + iT \), if \( \sigma < 1 \), and around \( s = -IT \), if \( \sigma < \beta \). The former, if present, will produce an integral contributing \( O(T^{-1}) \), and the latter, if present, will be

\[
\int_{-T}^{T} \frac{du}{u^2}.
\]

It follows that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{t + it}{r^2 + u^2} \right) dt = \sum \int_0^{\infty} \frac{du}{4 + (u + iT)^2} = O(1),
\]

for \( T > 1 \). We now make real parts and integrate for \( \frac{1}{2} < r < 2 \). Then by Theorem 13.9 we have

\[
\int \log \left( \frac{t + iT}{r^2 + u^2} \right) dt = \sum \int_0^{\infty} \left( \delta - \frac{1}{2} \right) R \left( \frac{1}{4 + (u + iT)^2} \right) du + O(1).
\]

By our hypothesis the integral on the left is \( O(\log T) \). Moreover

\[
R \left( \frac{1}{4 + (u + iT)^2} \right) \geq \left\{ \begin{array}{ll}
A (\sigma > 0) & \text{if } |T| \leq 1,
0, & \text{otherwise}.
\end{array} \right.
\]

If \( \sigma > \frac{1}{2} \) is given, then each zero counted by \( N(\sigma, T+1) - N(\sigma, T) \) contributes at least \( \frac{1}{2}(\sigma - \frac{1}{2})^2 \alpha \) to the sum on the right of (13.8.1), whence

\( N(\sigma, T+1) - N(\sigma, T) = o(\log T) \). Theorem 13.8 therefore follows from Theorem 13.5.
XIV
CONSEQUENCES OF THE RIEMANN HYPOTHESIS

14.1. In this chapter we assume the truth of the unproved Riemann hypothesis, that all the complex zeroes of \( \zeta(s) \) lie on the line \( s = \frac{1}{2} \). It will be seen that a perfectly coherent theory can be constructed on this basis, which perhaps gives some support to the view that the hypothesis is true. A proof of the hypothesis would make the 'theorems' of this chapter essential parts of the theory, and would make unnecessary much of the tentative analysis of the previous chapters.

The Riemann hypothesis, of course, leaves nothing more to be said about the 'horizontal' distribution of the zeroes. From it we can also deduce interesting consequences both about the 'vertical' distribution of the zeroes and about the order problems. In most cases we obtain much more precise results with the hypothesis than without it. But even a proof of the Riemann hypothesis would not by any means complete the theory. The finer shades in the behaviour of \( \zeta(s) \) would still not be completely determined.

On the Riemann hypothesis, the function \( \log(\zeta(s)) \), as well as \( \zeta(s) \), is regular for \( \sigma > \frac{1}{2} \) (except at \( s = 1 \)). This is the basis of most of the analysis of this chapter.

We shall not repeat the words 'on the Riemann hypothesis', which apply throughout the chapter.

14.2. Theorem 14.2.† We have

\[
\log(\zeta(s)) = O(\log t^{1-\delta+\epsilon}) \tag{14.2.1}
\]

uniformly for \( \frac{1}{2} < \sigma < \epsilon \leq 1 \).

Apply the Borel-Carathéodory theorem to the function \( \log(\zeta(s)) \) and the circles with centres \( s = \sigma + it \) and radii \( \frac{1}{2} - 2\delta \) and \( \frac{1}{2} - 3\delta \left( 0 < \delta < \frac{1}{4} \right) \). On the larger circle

\[
\Re(\log(\zeta(s))) = \log(1) < A \log t.
\]

Hence, on the smaller circle,

\[
\Re(\log(\zeta(s))) < \frac{3-29}{2} A \log t + \frac{3-53}{2} \log(2t+1) < A \log t \tag{14.2.2}
\]

† Littlewood (I).

14.2. CONSEQUENCES OF RIEMANN HYPOTHESIS

Now apply Hadamard's three-circles theorem to the circles \( C_1, C_2, C_3 \) with centres \( a_j = \epsilon_j \) \((1 < \epsilon_j < \epsilon)\), passing through the points \( 1+\gamma+it, \sigma+it, \frac{1}{2}+\delta+it \). The radii are thus

\[
r_j = a_j - \epsilon_j, \quad r_j = a_j - \frac{1}{2}, \quad r_j = a_j - \frac{1}{2} - \delta.
\]

If the maxima of \( \log(\zeta(s)) \) on the circles are \( M_j \), \( M_j \), \( M_j \), we obtain

\[
M_j \leq M_j + M_j,
\]

where

\[
a = \log t / \log \frac{5}{3} \approx \log \left( 1 + \frac{1+\frac{1+\frac{1+\frac{1+\frac{1+\frac{1}{2}}{1-\gamma}}{1-\gamma}}{1-\gamma}}{1-\gamma} \right) \]

\[
= \frac{1+\frac{1+\frac{1+\frac{1+\frac{1+\frac{1}{2}}{1-\gamma}}{1-\gamma}}{1-\gamma}}{1-\gamma}}{1+\frac{1}{2}+\delta + O(\frac{1}{\log t}) + O(\frac{1}{\log t}) + O(\frac{1}{\log t})}.
\]

By (14.2.2), \( M_j < A \delta^{-1} \log t \); and, since

\[
\log(\zeta(s)) = \sum_{n=1}^{\infty} \Lambda(n) / n^s \quad (\Lambda(n) \leq 1),
\]

\[
M_j < \max \left[ \sum_{n=1}^{\infty} \Lambda(n) / n^s \leq \sum_{n=1}^{\infty} 1 / n^{1+\frac{1}{2}} \right] \leq \frac{A}{\xi^{1+\frac{1}{2}}}.
\]

Hence

\[
[\log(\zeta(s))] \sim \left( A \frac{1}{\xi^{1+\frac{1}{2}}} \right) \log t \quad (\xi = \text{the order of } \log(\zeta(s)))
\]

The result stated follows on taking \( \delta \) and \( \eta \) small enough and \( \gamma \) large enough. More precisely, we can take

\[
\delta = \frac{1}{2} - \frac{1}{4} \quad \text{and} \quad \eta = \log t.
\]

etc., we obtain

\[
\log(\zeta(s)) = O(\log t^{2-\delta} \log(\log t)^{2-\delta}) \quad \left( 1 + \frac{1}{\log t} \leq a < 1 \right),
\]

(14.2.4)

Since the index of \( \log t \) in (14.2.1) is less than unity if \( t \) is small enough, it follows that (with a new \( a \))

\[
-\log t < \log(\zeta(s)) \leq t \log t \left( t > t_0 \right),
\]

i.e., we have both

\[
\zeta(s) = O(t^n),
\]

\[
\frac{1}{\zeta(s)} = O(t^{-n}),
\]

(14.2.6)

for every \( a > \frac{1}{2} \).

In particular, the truth of the Lindelöf hypothesis follows from that of the Riemann hypothesis.
It also follows that for every fixed \( \sigma > \frac{1}{2} \), as \( T \to \infty \)
\[
\int_{1}^{T} \frac{dt}{t^{\sigma+1/2}} = \frac{2\pi}{\zeta(2\sigma)}.
\]
For \( \sigma > 1 \) this follows from (7.1.2) and (1.2.7). For \( \frac{1}{2} < \sigma \leq 1 \) it follows from (14.2.6) and the analysis of §7.8, applied to \( \operatorname{Re}(s) \) instead of to \( \operatorname{Re}(\sigma) \).

14.3. The function \( v(\sigma) \).

For each \( \sigma > \frac{1}{2} \) we define \( v(\sigma) \) as the lower bound of numbers \( a \) such that
\[
\log \zeta(s) = 0 + O(\log a).
\]
It is clear from (14.2.3) that \( v(\sigma) \leq 0 \) for \( \sigma > 1 \); and from (14.2.2) that \( v(\sigma) \leq 1 \) for \( \frac{1}{2} < \sigma \leq 1 \); and in fact from (14.2.1) that \( v(\sigma) \leq 2 - 2\epsilon \) for \( \frac{1}{2} < \sigma \leq 1 \).

On the other hand, since \( \Lambda_1(2) = 1 \), (14.2.3) gives
\[
\left| \log \zeta(s) \right| \geq \frac{1}{2\pi} - \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s},
\]
and hence \( v(\sigma) \geq 0 \) if \( \sigma \) is so large that the right-hand side is positive. Since
\[
\sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+1/2}} = \frac{\pi}{\sin\pi\sigma}< \frac{1}{\sigma-1}
\]
this is certainly true for \( \sigma > 3 \). Hence \( v(\sigma) = 0 \) for \( \sigma > 3 \).

Now let \( \frac{1}{4} < c_1 < \sigma < c_2 \leq 4 \), and suppose that
\[
\log \zeta(s) = O(\log a), \quad \log \zeta(s) = O(\log a),
\]
where \( k(s) \) is the linear function of \( s \) such that \( k(c_1) = a, k(c_2) = b \), viz. \( k(s) = (x-c_1)(s-c_2) + (c_2-s)c_1 \).

Here
\[
\log(1-\epsilon)-\log(1-\epsilon) = \frac{\pi}{\sin\pi\sigma}
\]
denote the branches which are real for \( \sigma = 0 \). Thus
\[
\log(1-\epsilon) = \log(1+\epsilon) = \log(1+\epsilon) = \log(1+\epsilon) = \log(1+\epsilon) = \log(1+\epsilon) = \log(1+\epsilon) = O(1).
\]

We can also show that \( \zeta(s) \) has the same \( v(\sigma) \) as \( \log \zeta(s) \). Let \( v_1(\sigma) \) be the \( v(\sigma) \) of \( \zeta(s) \).

Hence
\[
\frac{v(\sigma)}{v_1(\sigma)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1+\epsilon)}{(x+\epsilon)^{1/2}} \, dx = O(\log(\log(1+\epsilon))).
\]

We have
\[
v_1(\sigma) \leq v(\sigma)
\]
for every positive \( \sigma \); and since \( v(\sigma) \) is continuous it follows that
\[
v_1(\sigma) \leq v(\sigma).
\]

We can show, as in the case of \( v(\sigma) \), that \( v_1(\sigma) \) is non-increasing, and is zero for \( \sigma \geq 3 \). Hence for \( \sigma < 3 \)
\[
\log \zeta(s) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1+\epsilon)}{(x+\epsilon)^{1/2}} \, dx = -O(\log(\log(1+\epsilon))),
\]
i.e.
\[
v_1(\sigma) \leq v(\sigma).
\]
The exact value of \( \zeta(s) \) is not known for any value of \( s \) less than 1.
All we know is

**Theorem 14.3.** For \( \frac{1}{2} < \sigma < 1 \),

\[
1 - \sigma < \zeta(s) < 2(1 - \sigma).
\]

The upper bound follows from Theorem 14.2 and the lower bound from Theorem 8.12. The same lower bound can, however, be obtained in another and in some respects simpler way, though this proof, unlike the former, depends essentially on the Riemann hypothesis. For the proof we require some new formulæ.

**14.4. Theorem 14.4.** As \( t \to \infty \),

\[
C(t) = \frac{\zeta(s)}{\zeta(t)} = e^{-\gamma t} + \frac{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}}{t^s} \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} t^{-s} + O(e^{-\gamma t}),
\]

uniformly for \( \frac{1}{2} < s < 1 \), \( e^{-\alpha} \leq s \leq \delta < 1 \).

Taking \( s = 1 \), \( f(t) = -\zeta'(s)/\zeta(s) \) in the lemma of § 7.9, we have

\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} x^{-s} = \frac{1}{2\pi i} \int_{x^{-1-s} = \epsilon} \frac{\zeta'(s)}{\zeta(s)} x^s ds,
\]

Now, by Theorem 9.6 (ii),

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{2\pi i}{\log x},
\]

and there are \( O(\log x) \) terms in the sum. Hence

\[
\frac{\zeta'(s)}{\zeta(s)} = O(\log t)
\]

on any line \( s \neq \frac{1}{2} \). Also

\[
\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\log x}{\log x - \gamma}\right) + O(\log t)
\]

uniformly for \(-1 < s < 2\). Since each interval \((n, n+1)\) contains values of \( s \) whose distance from the ordinate of any zero exceeds \( A/\log x \), there is a \( t_k \) in each interval for which

\[
\frac{\zeta'(s)}{\zeta(s)} = O(\log t)
\]

\(-1 < \sigma < 2, \ t = t_k\).

T Littlewood (6), to the end of § 14.8.

By the theorem of residues,

\[
\frac{1}{2\pi i} \int_{1 - it}^{1 + it} \frac{\zeta'(s)}{\zeta(s)} x^s ds = \frac{\zeta'(s)}{\zeta(s)} + \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} x^{-s} - \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} x^{1-s} + O(\log x),
\]

The integrals along the horizontal sides tend to zero as \( x \to \infty \), so that

\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} x^{1-s} = \frac{1}{2\pi i} \int_{1 - it}^{1 + it} \frac{\zeta'(s)}{\zeta(s)} x^{1-s} ds - \frac{\zeta'(s)}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} x^{1-s}.
\]

Since \( \Gamma(1-s) = O(e^{-\pi x/2}) \), the integral is

\[
O\left(\int_{-\infty}^{\infty} e^{-\pi x/2} \log(1+2\pi y^2) \, dy\right)
\]

and is

\[
O\left(\int_{-\infty}^{\infty} e^{-\pi x/2} \log(1+2\pi y^2) \, dy\right) - O\left(\int_{-\infty}^{\infty} e^{-\pi x/2} \log(1+2\pi y^2) \, dy\right).
\]

Also

\[
\Gamma(1-s) x^{-s} = O(e^{-\pi x (1-s)} - O(e^{-\pi x} - O(\log x)),
\]

\[
\Gamma(1-s) x^{-s} = O(e^{-\pi x (1-s)} - O(e^{-\pi x} - O(\log x)),
\]

This proves the theorem.

**14.5.** We can now prove more precise results about \( \zeta'(s)/\zeta(s) \) and \( \log \zeta(s) \) than those expressed by the inequality \( \sigma = -2 \). We have

\[
\frac{\zeta'(s)}{\zeta(s)} = O(\log t)^{1/4}
\]

\[
\log \zeta(s) = O(\log t)^{-1/2}
\]

uniformly for \( \frac{1}{2} < x \leq \sigma < 1 \).

We have

\[
\frac{\zeta'(s)}{\zeta(s)} \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\pi n} + \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} x^{-s} + \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} x^{-s}.
\]
Moving the line of integration in (14.4.3) to $R(w) = a$, we have
\[
\sum_{\gamma \in \gamma(w)} \frac{\Lambda(n)}{n^s} \rightarrow - \frac{\zeta'(s)}{\zeta(s)} - \Gamma(1-s) \psi^* - \frac{1}{2i\pi} \int \zeta(s) \frac{\zeta'(s)}{\zeta(s)} \psi^*(z) \frac{dz}{z^s}.
\]
Since $\zeta'/\zeta$ has the $\pi$-function $\pi(s)$, the integral is of the form
\[
O\left( \int_a^b e^{-\sigma t} \log^{(m+1)}(t) \frac{dt}{t} \right) = O\left( \sigma^m (\log t)^{m+1} \right);
\]
and $\Gamma(1-s)$ is also of this form, as in § 14.4. Hence
\[
\frac{\zeta'(s)}{\zeta(s)} = \sum_{\gamma \in \gamma(w)} \frac{\Lambda(n)}{n^s} \rightarrow \sum_{\gamma \in \gamma(w)} \frac{\Lambda(n)}{n^s} - \frac{1}{2i\pi} \int \frac{\zeta(s)}{\zeta(s)} \psi^*(z) \frac{dz}{z^s} = O\left( \sigma^m (\log t)^{m+1} \right).
\]
This result holds uniformly in the range $[a, b]$, and so we may integrate over this interval. We obtain
\[
\log \zeta(s) = \sum_{\gamma \in \gamma(w)} \frac{\Lambda(n)}{n^s} - \frac{1}{2i\pi} \int \frac{\zeta(s)}{\zeta(s)} \psi^*(z) \frac{dz}{z^s} = O(1),
\]
as required.

14.7. Proof that $\log \zeta(s)$ converges for $s > 1$. Theorem 14.6 enables us to extend the method of Davenport's approximation, already used for $\sigma > 1$, to values of $\sigma$ between $\frac{1}{2}$ and 1. It gives
\[
\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \log n e^{-2\pi n} + O\left( \sigma^m (\log t)^{m+1} \right) + O(1),
\]
whenever $t \leq 2^{10}$ and $n \leq 2^{10} t$. Theorem 14.6 is used for $\sigma > 1$, and we obtain $\psi^*(z) = 0$ for $z = 1$ as we please, the second result (with $\psi^*$ for $\psi$) follows. For fixed $\alpha$ and $\epsilon$ such that $\frac{1}{2} < \alpha < \epsilon < 1$, and $s = 1 - \epsilon < 1$,
\[
\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-2\pi n} + O\left( \sigma^m (\log t)^{m+1} \right) + O(1).
\]

Let us assume for the moment that this number $\epsilon$ satisfies the condition of Theorem 14.6 that $\epsilon < 1$. It gives
\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-2\pi n} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-2\pi n} + \sum_{q=1}^{\infty} \frac{2\pi n}{q} \Lambda(n) e^{-2\pi n} + O(1) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-2\pi n} + \sum_{q=1}^{\infty} \frac{2\pi n}{q} \Lambda(n) e^{-2\pi n} + O(1).
\]
Now
\[
\sum_{\mathfrak{q} \leq 1} \Delta_{\mathfrak{q}}(s) e^{\mathfrak{q}s} \leq \frac{1}{\log N} \sum_{\mathfrak{q} \leq 1} \Delta_{\mathfrak{q}}(s) e^{\mathfrak{q}s} + O\left(\sum_{\mathfrak{q} \geq N+1} e^{\mathfrak{q}s}\right) + O\left(\frac{\log N}{\delta}\right)
\]
as in §4.5. Hence
\[
\log \|\mathfrak{c}(s)\| \geq \frac{\mathfrak{c}(s) e^{-s} + O\left(\frac{\mathfrak{c}(s)}{\delta}\right)}{\log N} + \frac{\mathfrak{c}(s)}{\delta} + O\left(\frac{\mathfrak{c}(s) \log N}{\delta}\right) + O(\log N \log \log \varphi^2 N)
\]
Take \( q = N = [s^{-a}] \), where \( a > 1 \). The second and third terms on the right are then bounded. Also
\[
\log t \leq N \log q + 2 \log \varphi \leq \frac{a}{\log \varphi} \log \frac{1}{\delta} + \log 2\varphi,
\]
so that
\[
\delta \lesssim K(\log t)^{-\alpha-a},
\]
Hence
\[
\log \|\mathfrak{c}(s)\| > K(\log t)^{1-a} + O((\log t)^{1-a} \log \log \varphi^2 N),
\]
where \( \mathfrak{c} \) and \( \mathfrak{c}' \) are functions of \( a \) which tend to zero as \( a \to 1 \).

If the first term on the right is of larger order than the second, it follows at once that \( \mathfrak{c}(s) \geq 1 \). Otherwise
\[
\alpha - a + \mathfrak{c}(s) \geq 1 - a,
\]
and making \( a \to 1 \) the result again follows.

We have still to show that the \( t \) of the above argument satisfies \( e^{-t} \ll 5 \). Suppose on the contrary that \( t < e^{-t} \) for some arbitrarily small value of \( t \). Now, by (8.4.4),
\[
|\mathfrak{c}(s)| \geq \left| e^{-t} - 2N^{1-s} \right| \geq \frac{A}{\log N} \left( 1 - 2N^{1-s} \right)
\]
for \( \sigma > 1, \varphi \geq 6 \). Taking \( \varphi = 1 + \log \varphi \),
\[
|\mathfrak{c}(s)| \geq \frac{A}{\log N} \geq A \log \frac{1}{\delta} > A\delta.
\]
Since \( |\mathfrak{c}(s)| \to 0 \) and \( t \geq 2\varphi, \varphi \to \infty \), and the above result contradicts Theorem 3.3. This completes the proof.

14.8. The function \((1+i)t\). We are now in a position to obtain fairly precise information about this function. We shall first prove

Theorem 14.8. We have
\[
|\log \zeta(1+it)| \lesssim \log \log t + A. \tag{14.8.1}
\]

In particular
\[
\zeta(1+it) = O((\log \log t)^{1/2}), \tag{14.8.2}
\]
\[
\frac{1}{\zeta(1+it)} = O((\log \log t)^{1/2}). \tag{14.8.3}
\]

Taking \( \sigma = 1, \varphi = 1 \) in Theorem 14.6, we have
\[
|\log \zeta(1+i\varphi)| \lesssim \sum_{\mathfrak{q} \leq 1} \Delta_{\mathfrak{q}}(s) e^{\mathfrak{q}s} + O(\delta \log t) + O(1)
\]
\[
\lesssim \sum_{\mathfrak{q} \leq 1} \Delta_{\mathfrak{q}}(s) + \sum_{\mathfrak{q} \geq N+1} e^{\mathfrak{q}s} + O(\delta \log t) + O(1)
\]
\[
\lesssim \log N + O(e^{-\mathfrak{c}(s)} \delta) + O(\delta \log t) + O(1)
\]
by (3.14.4). Taking \( \delta = \log^2 t, N = 1 + [\log^2 t] \), the result follows.

Comparing this result with Theorems 8.6 and 8.8, we see that, as far as the order of the functions \( \zeta(1+it) \) and \( 1/\zeta(1+it) \) is concerned, the result is final. It remains to consider the values of the constants involved in the inequality.

14.9. We define a function \( \beta(\sigma) \) as
\[
\beta(\sigma) = \frac{\varphi(\sigma)}{\mathfrak{c}(\sigma)}.
\]
By the convexity of \( \varphi(\sigma) \) we have, for \( \frac{1}{2} \leq \sigma < \sigma' < 1, \)
\[
\varphi(\sigma') \leq \frac{1 - \sigma' \varphi(\sigma)}{1 - \sigma' \varphi(\sigma)} \leq \frac{1 - \sigma' \varphi(\sigma)}{1 - \sigma' \varphi(\sigma)}
\]
i.e.,
\[
\beta(\sigma') \leq \beta(\sigma).
\]
Thus \( \beta(\sigma) \) is non-increasing in \( \{\sigma, 1\} \). We write
\[
\beta(1) = \lim_{\sigma \to 1^+} \beta(\sigma), \quad \beta(1) = \lim_{\sigma \to 1^-} \beta(\sigma).
\]
Then by Theorem 14.3, for \( \frac{1}{2} \leq \sigma < 1, \)
\[
\frac{1}{2} \leq \beta(1) \leq \beta(\sigma) \leq \beta(1) \leq 1.
\]
We shall now prove

Theorem 14.9. As \( t \to \infty \)
\[
|\zeta(1+it)| \ll e^{(1+i)\varphi(1+o(1))\log t}, \tag{14.9.1}
\]
\[
\frac{1}{\mathfrak{c}(1+it)} \ll e^{(1+i)\varphi(1+o(1))\log t}. \tag{14.9.2}
\]

† Littlewood (19).
We observe that the $O(1)$ in Theorem 14.6 is actually $o(1)$ if $\delta \to 0$.

Also, taking $\sigma = 1$,

$$3^{1-\sigma} \log 2^{1+\sigma} = O(1)$$

if

$$\delta = (\log q)^{-\delta_0 - \eta} \quad (\eta > 0).$$

Hence, for such $\delta$,

$$\log \zeta(1+i\delta) = \sum_{\nu=1}^{\infty} A_{1}(n) e^{n \nu} + O(1)$$

$$= \sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1)$$

$$= \sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1).$$

Now the modulus of the second double sum does not exceed

$$\sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1).$$

This is evidently uniformly convergent for $\delta > 0$, the summand being less than $p^{-\nu}$. Since each term tends to zero with $\delta$ the sum is $o(1)$.

Hence

$$\log \zeta(1+i\delta) = \sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1)$$

$$= \sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1)$$

$$= \sum_{p \geq 2} \sum_{n \geq 1} e^{p \nu} + O(1).$$

The second term is $O(e^{-\delta}(\delta) = o(1)$ if $\sigma = [3^{-1} - 1]$. Also

$$1 - \frac{1}{p} < 1 - \frac{1}{p^{\nu}} < 1 + \frac{1}{p^{\nu}}.$$

Hence, by (3.15.3),

$$\log \zeta(1+i\delta) \leq \sum_{p \geq 2} \log \left(1 + \frac{1}{p^{\nu}}\right) + o(1)$$

$$= \log \zeta + \frac{1}{2} + o(1),$$

or

$$\log \zeta \leq \frac{1}{2} + o(1).$$

Now

$$\log \zeta \leq (1+\varepsilon) \log \frac{1}{\delta} = (1+\varepsilon) (\log \delta + \eta) \log \log \delta,$$

and taking a arbitrarily near to 1, we obtain (14.9.1). Similarly, by (3.15.3),

$$\log \frac{1}{\zeta(1+i\delta)} \leq \sum_{p \geq 2} \log \left(1 + \frac{1}{p^{\nu}}\right) + o(1)$$

$$= \log \log \sigma + \log \frac{1}{\delta} + o(1),$$

and (14.9.2) follows from this.

Comparing Theorem 14.9 with Theorems 8.9 (A) and (B), we see that, since we know only that $\beta(1) < 1$, in each problem a factor $\delta$ remains in doubt. It is possible that $\beta(1) = \varepsilon$, and if this were so each constant would be determined exactly.

14.10. The function $S(\delta)$. We shall now discuss the behaviour of this function on the Riemann hypothesis.

If $1 < a < \sigma < \beta$, $T < t < T$, we have

$$\log \zeta(s) = \int_{s-\delta}^{s+\delta} \log \zeta(s+i\tau) \frac{d\tau}{\varphi(s+i\tau)}.$$

Let $\beta > 2$. By (14.2.2),

$$\int_{s+\delta}^{s-\delta} \log \zeta(s+i\tau) \frac{d\tau}{\varphi(s+i\tau)} = O\left(\frac{1}{\varphi(s+i\tau)} \log \zeta(s+i\tau)\right) ds.$$
Also
\[ \int_{\sigma}^{T} \frac{\log \xi(z)}{z-\sigma} \, dz = -\frac{\log T}{\beta-\sigma}. \]
Making \( \beta \to \infty \), it follows that
\[ \log \xi(s) = \int_{\sigma}^{T} \frac{\log \zeta(1/2 + it)}{s-1/2 + it} \, dt + O(1). \]

A similar argument shows that, if \( \Re(s) < \sigma \),
\[ \int_{1/2}^{T} \frac{\log \zeta(1/2 + it)}{s-1/2 + it} \, dt = O(1). \]
Taking \( \sigma' = \sigma + 4 \), so that
\[ \sigma' - \varepsilon = \sigma - \varepsilon + 4 \approx \sigma - \varepsilon, \]
and replacing (14.10.5) by its conjugate, we have
\[ \int_{1/2}^{\infty} \frac{\log \zeta(1/2 + it)}{s-1/2 + it} \, dt + O(1). \]
From (14.10.1.1) and (14.10.3) it follows that
\[ \log \xi(s) = \frac{1}{\pi} \int_{1/2}^{\infty} \frac{\log \xi(z)}{z-\sigma} \, dz + O(1). \]
and
\[ \log \xi(s) = \frac{1}{\pi} \int_{1/2}^{\infty} \frac{\log \xi(z)}{z-\sigma} \, dz + O(1). \]

14.11. We can now show that each of the functions
\[ \max(\log \xi(s), 0), \quad \max(-\log \xi(s), 0), \]

\[ \max(\arg \xi(s), 0), \quad \max(-\arg \xi(s), 0) \]
has the same \( \zeta \)-function as \( \log \xi(s) \). Consider, for example,
\[ \max(\arg \xi(s), 0), \]
and let its \( \zeta \)-function be \( \nu(\sigma) \). Since
\[ |\arg \xi(s)| \leq \log |\xi(s)| \]
we have at once
\[ \nu(\sigma) \leq \nu(\sigma). \]
The above analysis shows that this is false if \( a < \frac{1}{2} \), which is satisfied if \( a < \frac{1}{4} \) and \( a \) is near enough to \( \frac{1}{2} \). This proves the first result, and the other may be proved similarly.

**Theorem 14.12 (3).**

\[ S_1(t) = O((\log t)^{1+\epsilon}). \]

From (14.10.5) with \( a = \frac{1}{4} \) we have

\[
\log \xi(t) - \int_0^1 \frac{S(y)}{y - \frac{1}{4} + iy} dy = O(1)
\]

\[
- \int \frac{S(y)}{(y - \frac{1}{4} + iy)^2} dy = O(1)
\]

\[
= \log \xi(t) = O(\log y),
\]

since \( S_1(y) = O(\log y) \). The result now follows as before.

In view of the result of Selberg stated in § 9.9, this theorem is true independently of the Riemann hypothesis. In the case of \( S(t) \), Selberg's method gives only an index \( \frac{1}{4} \) instead of the index \( \frac{1}{2} \) obtained on the Riemann hypothesis.

14.13. We now turn to results of the opposite kind.† We know that without any hypothesis

\[ S(t) = O(\log t), \quad S_1(t) = O(\log t), \]

and that on the Lindelöf hypothesis, and a fortiori on the Riemann hypothesis, each \( O \) can be replaced by \( o \). On the Riemann hypothesis we should expect something more precise. The result actually obtained is

**Theorem 14.13.**

\[
S(t) = O\left(\frac{\log t}{\log \log t}\right),
\]

\[
S_1(t) = O\left(\frac{\log t}{\log \log t}\right).
\]

We first prove three lemmas.

**Lemma a.** Let

\[ \psi(t) = \max \{ |S(u)| \}, \]

so that \( \psi(t) \) is non-decreasing, and \( \psi(t) = O(\log t) \). Then

\[ S(t) = O\left(\psi(2\log \log t)\right). \]

† Landau (11), Cramèr (1), Littlewood (4), Titchmarsh (3).

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14.13. **Consequences of Riemann Hypothesis.**

This is independent of the Riemann hypothesis. We have

\[ N(t) = L(t) + R(t), \]

where \( L(t) \) is defined by (1.3.1), and \( R(t) = N(t) + O(1/\beta) \). Now

\[ N(T + x) = N(T) + \beta > 0 \quad (0 < x < T). \]

Hence

\[ R(T + x) - R(T) = \beta + (L(T + x) - L(T)) > -Ax \log T. \]

Hence

\[ \int_{\frac{1}{2}}^1 R(t) dt = xR(T) + \int_{\frac{1}{2}}^1 (R(T + t) - R(T)) dt \]

\[ > xR(T) - \beta \int_{\frac{1}{2}}^1 \log T dt \]

\[ > xR(T) - Ax \log T. \]

Hence

\[ R(T) < \frac{1}{A} \int_{\frac{1}{2}}^1 R(t) dt + Ax \log T \]

\[ - S_1(T + x) - O\left(\frac{1}{T}\right) + Ax \log T \]

\[ = O\left(\frac{1}{T}\right) + Ax \log T. \]

Taking \( x := [2(T)/\log T] \), the upper bound for \( S(T) \) follows. Similarly by considering integrals over \((T - x, T)\) we obtain the lower bound.

**Lemma b.** Let \( 0 < t \leq 1 \), and let

\[ \tau(t) = \max \{ \log : \xi(t) : \log \xi(t) \}
\]

\[ \left( \begin{array}{c} \frac{1}{2} \log \log t \leq \sigma \leq \frac{1}{2} \log T \quad : \quad t \leq \tau(t) \end{array} \right). \]

Then

\[ \log \xi(t) = O\left(\frac{1}{\log \log T} \right) \]

\[ \left( \begin{array}{c} \frac{1}{2} \log \log t \leq \sigma \leq \frac{1}{2} \log T \quad : \quad t \leq \tau(t) \end{array} \right). \]

We apply Hadamard's three-circle theorem as in § 14.2, but now take

\[ \sigma_1 = \frac{3}{2} \log \log T, \quad \sigma_2 = 1 - \frac{1}{2} \log \log T, \quad \delta = \frac{1}{2} \log \log T, \quad \epsilon = \frac{1}{2}. \]

We obtain

\[ M_2 \leq M_1, \quad M_3 = M_1(1 - \delta)^{1 - \epsilon}, \]

where

\[ M_3 \leq \tau(T - 1), \]

and

\[ 1 - \alpha = \log \left( \frac{2}{\log \log 2} \right) = \log \left( 1 + \frac{1 - \frac{1}{2} - \frac{1}{2} \log \log 2}{\epsilon} \right) / \log \left( \frac{2}{\log \log 2} \right) \]

\[ > A (\sigma - \frac{1}{2} \log t). \]
Hence
\[ m < A \mathcal{F}(T+1)^{1-\delta} T^{-\frac{1}{2}+\frac{1}{6}+\frac{1}{8}-\frac{1}{2}} \cdot A \mathcal{F}(T+1)^{\frac{1}{2}+\frac{1}{6}+\frac{1}{8}-\frac{1}{2}}. \]

This gives the required result if \( \sigma \leq \frac{1}{2} \), and for \( \frac{1}{2} < \sigma < 2 \) it is trivial, if the \( A \) is small enough.

**Lemma γ.** For \( \sigma > \frac{1}{2} \), \( 0 < \xi < \frac{1}{2} \),
\[ \log \zeta(\xi) = \int_{\frac{1}{2}}^{\xi} S(t, \xi) dt + O(1). \] (14.13.3)

We have
\[ \int_{\frac{1}{2}}^{\xi} S(t, \xi) dt = \int_{\frac{1}{2}}^{\xi} \frac{S(t, \xi)}{t - \frac{1}{2}} dt + \int_{\frac{1}{2}}^{\xi} \frac{S(t, \xi)}{t - \frac{1}{2} - \frac{1}{2}} dt = O\left(\frac{S(\pi)}{\xi} + O(1)\right). \]

and similarly for the integral over \( (\xi, t - \frac{1}{2}] \). The result therefore follows from (14.12.4).

**Proof of Theorem 14.13.** By Lemmas α and γ,
\[ \log \zeta(\xi) = O(\phi(T) \log T) \int_{\frac{1}{2}}^{\xi} \frac{dy}{(y-\frac{1}{2})^2} + O(1) \]
\[ = O\left(\frac{\phi(T) \log T}{\xi} + O(1)\right) \]
for \( 0 < \xi \leq \frac{1}{2} \). Taking \( \xi = T \), we obtain
\[ \log \zeta(T) = O(T) \log T \int_{\frac{1}{2}}^{\xi} \frac{dy}{(y-\frac{1}{2})^2} + O(1). \]

Hence by Lemma β, for \( 0 < \xi \leq \frac{1}{2} \),
\[ \log \zeta(\xi) = O(T) \log T \int_{\frac{1}{2}}^{\xi} \frac{dy}{(y-\frac{1}{2})^2} + O(1) \]
for \( 0 < \xi \leq \frac{1}{2} \). Hence
\[ \int_{\frac{1}{2}}^{\xi} \log \zeta(\xi) d\xi = O(T) \log T \int_{\frac{1}{2}}^{\xi} \frac{dy}{(y-\frac{1}{2})^2} + O(1) \]
for \( 0 < \xi \leq \frac{1}{2} \). (14.13.4)

Again, the real part of (14.13.3) may be written
\[ \frac{\log \zeta(\xi)}{\xi} = \int_{\frac{1}{2}}^{\xi} \frac{x}{x-\frac{1}{2}} \frac{S(t, \xi)}{S(t+\xi)} dt + O(1). \]

Hence
\[ \frac{\log \zeta(\xi)}{\xi} = \int_{\frac{1}{2}}^{\xi} \frac{x}{x-\frac{1}{2}} \frac{S(t, \xi)}{S(t+\xi)} dt + O(1). \]

Now (14.13.4), (14.13.5), and Theorem 9.9 give
\[ \log \zeta(T) = O(T) \log T \int_{\frac{1}{2}}^{\xi} \frac{(\phi(T))}{\xi} + O(1). \]

Varying \( \xi \) and taking the maximum,
\[ \phi(T) = O(T) \log T \int_{\frac{1}{2}}^{\xi} \frac{(\phi(T))}{\xi} + O(1). \]

Let
\[ \psi(T) = \max_{T \leq t \leq T+1} \frac{\log T}{\phi(T)}, \]
so that \( \psi(T) \) is non-decreasing and
\[ \psi(T) \leq \frac{\log T}{\phi(T)} \phi(T). \]

Then (14.13.7) gives
\[ \phi(T) = O\left(\log T \log T \psi(T) \right). \]
or
\[ \phi(T) = O\left(\log T \log T \psi(T) \right). \]

Varying \( T \) and taking the maximum,
\[ \psi(T) = O\left(\log T \log T \psi(T) \right). \]

But \( \phi(T) \leq \frac{\phi(T)}{\psi(T)} \) for some arbitrarily large \( T \); for otherwise
\[ \psi(T) \leq \frac{\phi(T)}{\psi(T)} \]
and \( \phi(T) \leq \frac{\phi(T)}{\psi(T)} \). Hence
\[ \phi(T) \leq A(T). \]

i.e. \( \phi(T) \geq A(T) \) for some arbitrarily large \( T \), which is not so, since in fact \( \phi(T) \geq A(T) \). Hence
\[ \psi(T) \geq A(T). \]
for some arbitrarily large $T_0$, and so for all $T_0$, since $\psi$ is non-decreasing.

Hence

$$A(T) = O\left( \frac{\log T}{(\log \log T)^2} \right).$$

This proves (14.13.2), and (14.13.1) then follows from Lemma 9.6.

The argument can be extended to show that, if $S(h)$ is the zeta integral of $S(t)$, then

$$S(h) = o\left( \frac{\log t}{(\log \log t)^2} \right).$$

(14.13.8)

14.14. Theorem 14.13 also enables us to prove inequalities for $\zeta(s)$ in the immediate neighbourhood of $\sigma = \frac{1}{2}$, a region not touched by previous arguments. We obtain first

**Theorem 14.14 (A).**

$$\zeta(\frac{1}{2} + it) = O\left( \exp \left( A \log \frac{t}{(\log \log t)^2} \right) \right).$$

We have

$$S(t + x) - S(t) = \{N(t + x) - N(t)\} - \{(L(t + x) - L(t)) - (f(t + x) - f(t))\},$$

where $f(t)$ is the O(1) of (9.3.2), and arises from the asymptotic formula for $\log T(t)$. Thus $f(t) = O(1)^{\frac{1}{2}}$, and since $N(t + x) = N(t)$

$$S(t + x) - S(t) > -A \log t \cdot O(1)^{\frac{1}{2}} > -A \log t,$$

Hence, by (14.13.5),

$$\log |\zeta(s)| < A \int \frac{\sigma}{\log \log t} \left( \frac{\log t}{(\sigma - \frac{1}{2})^2 + \frac{1}{4}} \right) + O(1)$$

$$= A \int \frac{\sigma}{\log \log t} \left( \frac{\log t}{(\sigma - \frac{1}{2})^2 + \frac{1}{4}} \right) + O(1)$$

uniformly for $\sigma > \frac{1}{2}$, and so by continuity for $\sigma = \frac{1}{2}$. Taking

$$\xi = 1, \log t$$

the result follows.

**Theorem 14.14 (B).** We have

$$A \log t \left( \frac{\log t}{(\sigma - \frac{1}{2})^2 + \frac{1}{4}} \right) < \log |\zeta(s)| < A \log t \left( \frac{\log t}{(\sigma - \frac{1}{2})^2 + \frac{1}{4}} \right) + O(1),$$

(14.14.2)

$$\arg \zeta(s) = O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right),$$

(14.14.3)

By (14.13.1) and (14.13.3),

$$\log |\zeta(s)| = O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right) + O(1).$$


$$\int \frac{dx}{\sqrt{(\sigma - \frac{1}{2})^2 + \frac{1}{4}}} = \int \frac{dx}{\sqrt{1 + x^2}},$$

which is less than 1 if $\sigma < \frac{1}{2}$, and otherwise is less than

$$1 + \int \frac{dx}{\sqrt{x^2 + \frac{1}{4}}} = 1 + \log \frac{1}{\sigma - \frac{1}{2}}.$$

Taking $\xi = 1, \log t$, the lower bound in (14.14.2) follows. The upper bound follows from the argument of the previous section. Lastly, taking imaginary parts in (14.13.3),

$$\arg \zeta(s) = \int \frac{\sigma - \frac{1}{2}}{\sqrt{(\sigma - \frac{1}{2})^2 + \frac{1}{4}}} S(t + x) - S(t) \ dx +$$

$$= O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right) + O(1)$$

$$= O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right) + O(1)$$

$$= O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right) + O(1).$$

Now

$$\int \frac{\sigma - \frac{1}{2}}{\sqrt{(\sigma - \frac{1}{2})^2 + \frac{1}{4}}} \ dx < \int \frac{\sigma - \frac{1}{2}}{\sqrt{(\sigma - \frac{1}{2})^2 + \frac{1}{4}}} \ dx - \frac{\pi}{2}. $$

Hence, taking $\xi = 1, \log t$, uniformly for $\sigma > \frac{1}{2}$, and so by continuity for $\sigma = \frac{1}{2}$.

In particular

$$\log |\zeta(s)| = O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right),$$

(14.14.4)

From (14.14.4), (14.5.2), and a Phragmèn–Lindelöf argument it follows that

$$\log |\zeta(s)| = O\left( \frac{\log t}{(\log \log t)^{\frac{1}{2}}} \right),$$

(14.14.5)

uniformly for $\frac{1}{2} + A \log t \leq \sigma \leq 1 - A$.}

14.15. Another result in the same order of ideas is an approximate formula for $\log \zeta(s)$, which should be compared with Theorem 9.6 (B).

**Theorem 14.15.** For $\frac{1}{2} < \sigma < 1$,

$$\log \zeta(s) = \sum_{\nu \neq \rho} \frac{1}{\nu - s} + O\left( \frac{\log t \log \log t}{\log \log t} \right),$$

(14.15.1)
In Lemma a of § 3.9, let

$$f(s) = \zeta(s), \quad a_n = \frac{1}{s} + \frac{1}{s^2} + \frac{s}{2} T, \quad r = \frac{1}{s^2}.$$

where $\delta = 1/\log T$. By (14.14.4)

$$\frac{1}{\psi(r)} \leq \exp\left(\frac{A \log T}{\log T}\right)$$

The upper bound in (14.14.2) gives

$$|\zeta(s)| < \exp\left(\frac{A \log T}{\log T}\right)$$

for $|s-a_n| < r$, $\alpha > \frac{1}{2}$, and for $|s-a_n| < r$, $\alpha < \frac{1}{2}$, the functional equation gives

$$|\zeta(s)| < \exp\left|\frac{A \log T}{\log T}\right| < \exp\left(\frac{A \log T}{\log T}\right) < \exp\left(\frac{A \log T}{\log T}\right)$$

It therefore follows from (3.9.1) that

$$\log|\zeta(s)| - \log|\zeta(s)| - \sum_{n=1}^{\infty} \log|\zeta(s-a_n)| + \sum_{n=1}^{\infty} \log|\zeta(s-a_n)| = O\left(\frac{\log T}{\log T}\right)$$

for $|s-a_n| < \frac{1}{2}r$, and so in particular for $\frac{1}{2} \leq |s| \leq \frac{1}{2} + \delta$, $T = T$.

Now

$$\log|\zeta(s)| = O\left(\frac{\log T}{\log T}\right)$$

Also

$$a_n - r = \frac{1}{s^2} + \frac{s}{2} T - \gamma T$$

Hence

$$\frac{1}{s^2} \leq |s-a_n| < \delta$$

and so, if the logarithm has its principal value,

$$\log|\zeta(s)| = O\left(\frac{1}{s^2}\right) = O\left(\frac{\log T}{\log T}\right).$$

Also the number of values of $s$ in the above sums does not exceed

$$N\left(T; \frac{1}{s^2}\right) \approx N\left(T; \frac{1}{s^2}\right) = O(\log T)$$

by Theorem 14.13. Hence

$$\sum_{n=1}^{\infty} \log|\zeta(s-a_n)| = O\left(\frac{\log T}{\log T}\right)$$

and, as in the case of the other sums.

This proves the theorem.

For $\zeta(s)/\zeta(s)$ we obtain similarly from Lemma a of § 3.9

$$\frac{\zeta(1-s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{s-a_n} = O\left(\frac{\log T}{\log T}\right).$$

(14.15.2)

14.16. Theorem 14.16. Each interval $[T, T+1]$ contains a value of $s$ such that

$$|\zeta(s)| > \exp\left(-A \log T\right)\left(\frac{1}{2} \leq \alpha \leq 2\right).$$

Let $\delta = 1/\log T$. Then the lower bound (14.16.1) holds automatically for $\alpha > \frac{1}{2} + \delta$, by (14.4.3). We therefore assume that $\frac{1}{2} \leq \alpha \leq \frac{1}{2} + \delta + \gamma$.

If $\gamma - \frac{1}{2} = s$ and $s = \frac{1}{2} + \delta + \gamma$, then, as in (14.4.5), we find

$$\log|\zeta(s)| = \sum_{n=1}^{\infty} \log|\zeta(s-a_n)| + O\left(\frac{\log T}{\log T}\right).$$

Moreover log $|\zeta(s)| = O\left(\frac{\log T}{\log T}\right)$ by (14.14.4) so that, on taking real parts

$$\log|\zeta(s)| = \sum_{n=1}^{\infty} \frac{1}{s-a_n} = O\left(\frac{\log T}{\log T}\right)$$

since $|s-a_n| \geq |\gamma - 1|$. We now observe that

$$\int_{T}^{T+1} \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{s-a_n} = \int_{T}^{T+1} \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{s-a_n} \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{s-a_n} \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{s-a_n}$$

by Theorem 14.13. Hence

$$\sum_{n=1}^{\infty} \log|\zeta(s-a_n)| = O\left(\frac{\log T}{\log T}\right)$$

and, as in the case of the other sums.
as there are \(O(\log T)\) terms in the sum. Hence there is a \(t\) for which
\[
\sum_{n \leq x} \log \left(1 - \frac{t}{n} \right) \geq - A \log T
\]
and the result follows.

In particular, if \(\epsilon\) is any positive number, each \((T, T+1)\) contains a \(t\) such that
\[
\frac{1}{\zeta(t)} = O(\epsilon) \quad (1 \leq \epsilon \leq 2).
\]

14.17. Mean-value theorems† for \(S(t)\) and \(S(t)\). We consider first \(S(t)\). We begin by proving

**Theorem 14.17.** For \(\frac{1}{2} T \leq t \leq T, \delta - T^{-1} \leq \gamma \leq \delta,\)

\[
S(t) = C - \sum_{n=1}^{\infty} \frac{\Lambda(n) \cos(\log n)}{n! \log n} e^{-\gamma n} + O\left(\frac{1}{\log \log T}\right)
\]

where
\[
C = \int_{1}^{T} |\log \zeta(\sigma)| \, ds.
\]

Making \(\delta \to \infty\) in (9.9.4), we have

\[
S(t) = C - \int_{1}^{T} \log |\zeta(\sigma)| \, ds.
\]

Now, integrating (14.4.1) from \(s = \frac{1}{2} + it\),

\[
\log \xi(s) - \log \xi(\frac{1}{2} + it) = \sum_{n=1}^{\infty} \frac{\Lambda(n) e^{-\gamma n}}{n! \log n} (1 - e^{-\gamma n}) + \sum_{n=1}^{\infty} \frac{\Lambda(n) e^{-\gamma n}}{n! \log n} (1 - e^{-\gamma n})
\]

Also, if \(\epsilon \geq \frac{1}{2},\)

\[
\log |\zeta(\sigma)| - \log |\zeta(\frac{1}{2} + it)| = \sum_{n=1}^{\infty} \frac{\Lambda(n) (1 - e^{-\gamma n})}{n! \log n}
\]

Also, if \(\epsilon \geq \frac{1}{2},\)

\[
\log |\zeta(\sigma)| - \log |\zeta(\frac{1}{2} + it)| = \sum_{n=1}^{\infty} \frac{\Lambda(n) (1 - e^{-\gamma n})}{n! \log n}
\]

Also, if \(\epsilon \geq \frac{1}{2},\)

\[
\log |\zeta(\sigma)| - \log |\zeta(\frac{1}{2} + it)| = \sum_{n=1}^{\infty} \frac{\Lambda(n) (1 - e^{-\gamma n})}{n! \log n}
\]

Hence, for \(\frac{1}{2} \leq \epsilon \leq 1,\)

\[
\log |\zeta(\sigma)| - \log |\zeta(\frac{1}{2} + it)| = \sum_{n=1}^{\infty} \frac{\Lambda(n) (1 - e^{-\gamma n})}{n! \log n}
\]

† Littlewood (9), Titchmarsh (2).
Now
\[ \int_{E(x; \frac{1}{2})}^{\sqrt{x}} \frac{t^{\frac{1}{2}} - 1}{t} \, dt = \log \log \log x. \]

Also, by (14.13.1), for \( t + 1 \leq \gamma \leq t + 1 \)
\[ N(t + 1, \log \log t) - N(t) = O(\log \log t). \]

Hence
\[ \sum_{\gamma \in \Lambda(n)} \frac{1}{\gamma} = \int_{\gamma \in \Lambda(n)} \frac{1}{\gamma} \, d\gamma = O(\log \log t). \]

and
\[ \sum_{\gamma \in \Lambda(n)} \frac{1}{\gamma} = \sum_{\gamma \in \Lambda(n)} \frac{1}{\gamma} = \sum_{\gamma \in \Lambda(n)} \frac{1}{\gamma} = O(1). \]

Therefore, for the given \( \delta \) and \( \varepsilon \), this proves the theorem.

14.18. Lemma 14.16 If \( \alpha_n = O(1) \), \( \varepsilon < 1 \), then
\[ \frac{2}{T} \int_{\frac{1}{T}}^{\frac{1}{T}} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \, d\gamma = \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} + O(\frac{1}{\log T}). \]

Uniformly for \( \delta > 1 \). Similarly, if \( \alpha_n = 0(\log n) \), the formula holds with a remainder term
\[ O(\frac{1}{\log T}). \]

The left-hand side is
\[ \sum_{n=1}^{\infty} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \cdot \frac{1}{\gamma} \, d\gamma = \sum_{n=1}^{\infty} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \cdot \frac{1}{\gamma} \, d\gamma. \]

Clearly
\[ \sum_{n=1}^{\infty} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \cdot \frac{1}{\gamma} \, d\gamma = O\left( \frac{1}{\log T} \right) \sum_{n=1}^{\infty} e^{-\alpha_n \gamma}. \]

Also
\[ \sum_{n=1}^{\infty} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \cdot \frac{1}{\gamma} \, d\gamma = O\left( \frac{1}{\log T} \right) \sum_{n=1}^{\infty} e^{-\alpha_n \gamma}. \]

Hence
\[ \sum_{n=1}^{\infty} \sum_{\gamma \in \Lambda(n)} e^{-\alpha_n \gamma} \cdot \frac{1}{\gamma} \, d\gamma = O\left( \frac{1}{\log T} \right) \sum_{n=1}^{\infty} e^{-\alpha_n \gamma}. \]
and, since \( \int_0^1 (f(t) + t) dt \leq \int_0^1 \int_0^t f(t) dt + \frac{1}{2} \int_0^1 f(t)^2 dt \), it follows that

\[
\frac{1}{T} \int_0^T |S(t)|^2 dt = \frac{C}{\pi^2} + \frac{1}{2} \sum_{n=1}^\infty \frac{\Lambda(n)}{\log n} + O\left( \frac{1}{\log \log T} \right).
\]

Replacing \( T \) by \( \frac{1}{2}, \frac{1}{3}, \ldots \) and adding, we obtain the result.

14.20. The corresponding problem involving \( S(t) \) is naturally much more difficult, but it has recently been solved by A. Selberg (4). The solution depends on the following formula for \( \zeta(s) \):

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \left( \frac{\log z}{z} \right)^s + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s} + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+2)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s},
\]

where

\[
\Lambda(n) - \Lambda(s) \quad (1 < n < z), \quad \frac{\Lambda(n)}{\log z} \quad (z < n < z^2).
\]

Let \( n = \max(1, 1 + o) \). Then

\[
\frac{1}{2\pi i} \int_{n}^{n+1} \frac{z^{s-1} - z^{-1}}{z(s-1)} \frac{\zeta'(s)}{\zeta(s)} ds = \frac{1}{2\pi i} \sum_{n=1}^\infty \frac{\Lambda(n)}{n^{s-1}} \int_0^1 \left( e^{2\pi \gamma} - 1 \right) z^{-1} \frac{dz}{z(s-1)}
\]

\[
= -\frac{1}{2\pi i} \sum_{n=1}^\infty \frac{\Lambda(n)}{n^{s-1}} \int_0^1 \left( e^{2\pi \gamma} - 1 \right) z^{-1} \frac{dz}{z(s-1)}
\]

\[
= -\frac{1}{2\pi i} \sum_{n=1}^\infty \frac{\Lambda(n)}{n^{s-1}} \int_0^1 \left( e^{2\pi \gamma} - 1 \right) z^{-1} \frac{dz}{z(s-1)}
\]

\[
= -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{s-1}} \left( \log z - \log z^{1/2} \right) - \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \left( \log z - \log z^{1/2} \right)
\]

\[
= \log z \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s},
\]

14.20. CONSEQUENCES OF RIEMANN HYPOTHESIS

Now consider the residues obtained by moving the line of integration to the left. The residue at \( z = 1 \) is

\[
\frac{\zeta'(s)}{\zeta(s)} \left( \frac{\log z}{z} \right)^s + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+2)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s},
\]

those at \( z = \beta \) and \( z = \rho \) are

\[
\frac{\zeta'(s)}{\zeta(s)} \left( \frac{\log z}{z} \right)^s + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+2)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s},
\]

respectively. The result now easily follows.

14.21. Theorem 14.21. For \( \gamma > 2, 4 < x < x^2 \),

\[
\alpha_1 = \frac{1}{2} \frac{1}{\log z},
\]

we have

\[
S(t) = \epsilon - \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \sin(\log n) + O\left( \frac{1}{\log n} \right) \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \log n + O(\log x).
\]

By the previous theorem,

\[
\frac{\zeta'(s)}{\zeta(s)} \left( \frac{\log z}{z} \right)^s + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+2)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s},
\]

for \( s > \alpha_1 \), where \( |x| < 1 \). Now

\[
\frac{e^{2\pi \gamma} - 1}{e^{2\pi \gamma} - 1} \frac{1}{\log x} \leq \frac{e^{2\pi \gamma} - 1}{e^{2\pi \gamma} - 1} \frac{1}{\log z} - 2s x^{1/2} - 
\]

\[
-\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \log n + \log x.
\]

Hence

\[
\frac{\zeta'(s)}{\zeta(s)} \left( \frac{\log z}{z} \right)^s + \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+2)^s} - \frac{1}{\log z} \sum_{n=1}^\infty \frac{\Lambda(n)}{(2n+1)^s},
\]

Now by (2.12.7)

\[
\frac{\zeta'(s)}{\zeta(s)} = \epsilon - \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \log n - \frac{1}{\alpha_1 - \rho} - O(\log t).
\]

Hence

\[
\Re \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \log n - \frac{1}{\alpha_1 - \rho} + O(\log t),
\]

\[
\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \log n - \frac{1}{\alpha_1 - \rho} + O(\log t).
\]
Taking real parts in (14.21.2), substituting this on the left, and taking \(\sigma = \sigma_1\),

\[
\sum_{\gamma} \frac{-\frac{1}{2} - \frac{1}{2} \mathbf{i}}{(t^2)} + O(\log t) = -R \sum_{\gamma} \frac{\Lambda(t)}{\gamma} + \frac{2\omega}{e} \sum_{\gamma} \frac{-\frac{1}{2}}{(t^2)} + O(\log t).
\]

Here

\[
\left(1 - \frac{2\omega}{e}\right) \sum_{\gamma} \frac{-\frac{1}{2}}{(t^2)} + O(\log t) = -R \sum_{\gamma} \frac{\Lambda(t)}{\gamma} + O(\log t).
\]

Hence

\[
1 - \frac{2\omega}{e} > 1 - \frac{1}{2} > 1.
\]

Hence

\[
\sum_{\gamma} \frac{-\frac{1}{2}}{(t^2)} = 0, \quad \sum_{\gamma} \Lambda(t) = O(\log t). \tag{14.21.3}
\]

Inserting this in (14.21.2), we get

\[
\mathcal{U}(t + \mathbf{i}) \sum_{\gamma} \frac{\Lambda(t)}{\gamma} = O(\log t).
\]

Now

\[
\text{arg} \left(\frac{\mathcal{U}(t + \mathbf{i})}{\mathcal{U}(t)}\right) = -\int_{t}^{\infty} \frac{\mathcal{U}(s + \mathbf{i})}{\mathcal{U}(s)} ds + \frac{\tau}{\mathcal{U}(t)} \int_{t}^{\infty} \mathcal{U}(s + \mathbf{i}) ds - \frac{\mathcal{U}(t)}{\mathcal{U}(t + \mathbf{i})} \int_{t}^{\infty} \mathcal{U}(s + \mathbf{i}) ds + \frac{\mathcal{U}(t + \mathbf{i})}{\mathcal{U}(t)} \int_{t}^{\infty} \mathcal{U}(s + \mathbf{i}) ds + \frac{\mathcal{U}(t)}{\mathcal{U}(t)} \int_{t}^{\infty} \mathcal{U}(s + \mathbf{i}) ds.
\]

By (14.21.4)

\[
\int_{t}^{\infty} \sum_{\gamma} \frac{\Lambda(t)}{\gamma} ds + O(\log t) = \int_{t}^{\infty} \sum_{\gamma} \frac{\Lambda(t)}{\gamma} ds + O(\log t).
\]

Also, by (14.21.4) with \(\sigma = \sigma_1\),

\[
\left|\mathcal{U}(t + \mathbf{i}) \sum_{\gamma} \frac{\Lambda(t)}{\gamma}\right| = O(\log t).
\]

It remains to estimate \(J_1\). For \(\frac{1}{2} < \sigma < \sigma_1\),

\[
\left|\mathcal{U}(t + \mathbf{i}) \sum_{\gamma} \frac{\Lambda(t)}{\gamma}\right| = \sum_{\gamma} \left|\mathcal{U}(t + \mathbf{i}) \sum_{\gamma} \frac{\Lambda(t)}{\gamma}\right| + O(\log t)
\]

Hence

\[
\left(\frac{1}{\mathcal{U}(t)} \sum_{\gamma} \frac{\Lambda(t)}{\gamma}\right) \mathcal{U}(t + \mathbf{i}) \sum_{\gamma} \frac{\Lambda(t)}{\gamma} = O(\log t).
\]

by (14.21.3). The theorem follows from these results.

Theorem 14.11 leads to an alternative proof of Theorem 14.13; for taking \(x = \log t\) we obtain

\[
S(t) = O\left(\frac{1}{\log t} + \frac{1}{\log \log t} + O\left(\frac{1}{\log \log t}\right)\right).
\]
14.22. Theorem 14.22. For
\[ T^a < x \leq T^b \quad (0 < a \leq b) \]
\[ \int_{T^a}^{T^b} \left[ S(t) + \frac{1}{x} \sum_{p \leq x} \frac{\sin(\log t p)}{\log p} \right]^2 dt = O(T), \]
We have
\[ S(t) + \frac{1}{x} \sum_{p \leq x} \frac{\sin(\log t p)}{\log p} = \frac{1}{x} \sum_{p \leq x} \frac{\Lambda(p)-\Lambda(p)\rho^{\log p}}{\log p} + \]
\[ + O\left( \frac{1}{\log x} \left| \sum_{p \leq x} \frac{\Lambda(p)}{p} \right| \right) + O\left( \frac{1}{\log x} \left| \sum_{p \leq x} \frac{\Lambda(p)}{p} \right| \right) + \]
\[ + O\left( \frac{1}{\log x} \left| \sum_{p \mid 1} \frac{\Lambda(p)}{p} \right| \right) + O\left( \frac{1}{\log x} \left| \sum_{p \mid 1} \frac{1}{p} \right| \right) + O\left( \frac{\log x}{\log e} \right). \]
The last term is bounded if \( \frac{1}{2} T < t < T \), \( x > T^a \), where \( a \) is a fixed positive constant. The last term but one is
\[ O\left( \sum_{p \leq x} \frac{1}{p^b} \right) = O\left( \sum_{p \leq x} \frac{1}{p-1} \right) = O(1). \]
Now consider the first term on the right. If \( p \leq x \),
\[ \Lambda(p) - \Lambda(p)\rho^{\log p} = (1 - (1/p)\log p), \]
and, if \( x < p < x^a \), it is
\[ O\left( \frac{1}{\log x} \left| \sum_{p \leq x} \frac{\Lambda(p)}{p} \right| \right) + O\left( \frac{1}{\log x} \left| \sum_{p \mid 1} \frac{\Lambda(p)}{p} \right| \right) + O\left( \frac{\log x}{\log e} \right). \]
Hence the first term is the imaginary part of
\[ \sum_{p \leq x} \frac{\Lambda(p) - \Lambda(p)\rho^{\log p}}{\log p}, \]
where \( \sigma_p = \frac{\Lambda(p) - \Lambda(p)\rho^{\log p}}{\log p} = O(\rho^{\log p}/\log x). \)
Now
\[ \int_{T^a}^{T^b} \left| \sum_{p \leq x} \frac{\Lambda(p) - \Lambda(p)\rho^{\log p}}{\log p} \right|^2 \frac{1}{\log p} \frac{dt}{T^a} \]
\[ = O\left( \sum_{p \leq x} \frac{1}{p^2} \right) + O\left( \sum_{p \leq x} \frac{1}{p^2} \right) + O\left( \frac{\log x}{\log e} \right). \]

14.23. Theorem 14.23. If \( T^a < x < T^b \),
\[ \int_{T^a}^{T^b} \left[ \sum_{p \leq x} \frac{\sin(\log t p)}{\log p} \right]^2 \frac{1}{T^a} \frac{dt}{T^a} = \frac{1}{T^a} \log \log T + O(T). \]
This is
\[ \sum \sum \frac{1}{p^{2} \log p} \left[ \int \frac{\sin(t \log p)}{t} \sin(t \log p) \, dt \right] \]
\[ - \sum \sum \frac{1}{p^{2} \log p} \left[ \int \left( \frac{1}{\log p} \right) \, dt \right] - \left( \sum \sum \frac{1}{p^{2} \log p} \right) \]
Now, by (3.14.5),
\[ \sum \frac{1}{p^{2} \log p} = \log \log x + O(1) - \log \log x + O(1) \]
and (since \( p_{x} > A \log x \))
\[ \sum \frac{1}{p^{2} \log p} = O(1). \]
Hence the first term is
\[ \frac{1}{T \log \log T + O(T)}. \]
Also \( \log p/q \geq A/p > A/p^{2} \).
Hence the remainder is
\[ O\left( \sum \sum \frac{1}{p^{2} \log p} \right) = O(x^{2} \log x) = O(x^{2}), \]
and the result follows if \( x < T \).

\[ \int_{\delta}^{T} (S(t))^{2} \, dt \leq \frac{1}{2 \pi} T \log \log T. \]
\[ \int_{\delta}^{T} (S(t))^{2} \, dt = \int_{\delta}^{T} \left( S(1) + \sum \frac{\sin(t \log p)}{\log p} \right)^{2} \, dt \]
\[ = \frac{1}{2 \pi} \int_{\delta}^{T} \left( S(1) + \sum \frac{\sin(t \log p)}{\log p} \right)^{2} \, dt \]
\[ - \frac{1}{\pi} \int_{\delta}^{T} \left( S(1) + \sum \frac{\sin(t \log p)}{\log p} \right) \left( \sum \frac{\sin(t \log p)}{\log p} \right) \, dt + \]
\[ \frac{1}{\pi} \int_{\delta}^{T} \left( \sum \frac{\sin(t \log p)}{\log p} \right)^{2} \, dt \]
\[ = O(T) + O(T \log \log T) + \frac{1}{2 \pi} T \log \log T + O(1) \]
(\text{using Schwarz's inequality on the middle term). The result then follows by addition.}

It can be proved in a similar way that
\[ \int_{\delta}^{T} (S(t))^{2} \, dt \sim \frac{2 \delta}{(2 \pi)^{2}} T^{2} \log \log T \]
for every positive integer \( k \).

14.25. The Dirichlet series for \( 1/\zeta(s) \). It was proved in § 3.13 that the formula
\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} \]
which is elementary for \( s > 1 \), holds also for \( s = 1 \). On the Riemann hypothesis we can go much farther than this.†

Theorem 14.25 (A). The series
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} \]  
(14.25.1)
is convergent, and its sum is \( 1/\zeta(s) \), for every \( s \) with \( s > -1/2 \).

In the lemma of § 3.12, take \( a_{n} = \mu(n), f(u) = 1/\zeta(u) \), \( s = 2 \), and \( \pi \) half an odd integer. We obtain
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}} = \frac{1}{2 \pi} \int_{\delta}^{T} \frac{1}{\zeta(s + w)} \, dw + O\left( \frac{1}{T} \right) \]
\[ = \frac{1}{2 \pi} \left( \int_{\delta}^{1-\delta} \frac{1}{\zeta(s + w)} \, dw + \int_{1-\delta}^{1+\delta} \frac{1}{\zeta(s + w)} \, dw + \frac{\log T}{\log \log T} + \frac{1}{\log \log T} \right) \]
\[ = \frac{1}{2 \pi} \left( \int_{\delta}^{1-\delta} \frac{1}{\zeta(s + w)} \, dw + \frac{\log T}{\log \log T} + \frac{1}{\log \log T} \right) \]
where \( 0 < \delta < s - 1/2 \).

By (14.2.5), the first and third integrals are
\[ O\left( \frac{1}{2 \pi} \int_{\delta}^{1-\delta} \, dw \right) = O(T^{-1/2 + \epsilon}), \]
and the second integral is
\[ O\left( \frac{1}{2 \pi} \int_{\delta}^{1+\delta} \, dw \right) = O(T^{-1/2 + \epsilon}). \]
Hence
\[ \sum_{n \in \mathbb{Z}} \frac{a(n)}{n^s} = \frac{1}{\zeta(s)} + \mathcal{O}(T^{-\varepsilon_1 + \varepsilon_2}) + \mathcal{O}(T^{1 - \frac{1}{2} + \varepsilon}). \]

Taking, for example, \( T = x^2 \), the \( O \)-terms tend to zero as \( x \to \infty \), and the result follows.

Conversely, if (14.25.1) is convergent for \( \sigma > \frac{1}{2} \), it is uniformly convergent for \( \sigma > \frac{1}{2} \), and so in this region represents an analytic function, which is \( 1/\zeta(s) \) for \( \sigma > 1 \) and so throughout the region. Hence the Riemann hypothesis is true. We have in fact

**Theorem 14.25 (B).** The convergence of (14.25.1) for \( \sigma > \frac{1}{2} \) is a necessary and sufficient condition for the truth of the Riemann hypothesis.

We shall write
\[ M(x) = \sum_{n \leq x} \mu(n). \]

Then we also have

**Theorem 14.25 (C).** A necessary and sufficient condition for the Riemann hypothesis is
\[ M(x) = O(x^{1 - \varepsilon}). \]

The lemma of § 3.12 with \( \varepsilon = 0 \), \( x \) half an odd integer, gives
\[
M(x) = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} \frac{1}{z(z - x)} \frac{dz}{(z)} + \mathcal{O}(x^{1 - 2\varepsilon}).
\]

By (14.2.6). Taking \( T = x^2 \), (14.25.2) follows if \( x \) is half an odd integer, and so generally.

Conversely, if (14.25.2) holds, then by partial summation (14.25.1) converges for \( \sigma > \frac{1}{2} \), and the Riemann hypothesis follows.

**Theorem 14.26.** The finer theory of \( M(x) \) is extremely obscure, and the results are not nearly so precise as the corresponding ones in the prime-number problem. The best \( O \)-result known is

**Theorem 14.26 (A).**
\[
M(x) = O\left( \frac{x \exp\left( \frac{\log x}{\log \log x} \right)}{\log x} \right).
\]

To prove this, take
\[
T = x^{1 - \varepsilon},
\]

in the formula (14.25.3). By (14.14.5),
\[
\left| \frac{1}{\zeta(s)} \right| \leq \exp\left( \frac{\log T}{\log \log T} \right)
\]
on the horizontal sides of the contour. The contribution of these is therefore
\[
\mathcal{O}\left( \frac{1}{T} \exp\left( \frac{\log T}{\log \log T} \right) \right) = O\left( \exp\left( \frac{\log x}{\log \log x} \right) \right).
\]

On the vertical side, (14.14.5) gives
\[
\left| \frac{1}{\zeta(s)} \right| \leq \exp\left( \frac{\log v}{\log \log v} \log \frac{2\log T}{\log \log v} \right)
\]

for \( v < x < T \). Now it is easily seen that the right-hand side is a steadily increasing function of \( v \) in this interval. Hence
\[
\left| \frac{1}{\zeta(s)} \right| \leq \exp\left( \frac{\log x}{\log \log x} \right) \quad (v < x < T).
\]

Hence the integral along the vertical side is of the form
\[
O(x^{1 - \varepsilon}) + \mathcal{O}(x^{1 - \varepsilon} \exp\left( \frac{\log T}{\log \log T} \right) \frac{dx}{x})
\]

\[ = O(x^{1 - \varepsilon} \exp\left( \frac{\log T}{\log \log T} \right) \log T) = O\left( \frac{x^{1 - \varepsilon} \exp\left( \frac{\log x}{\log \log x} \right)}{\log x} \right). \]

This proves the theorem.

**Theorem 14.26 (B).**
\[ M(x) = O(x^{1 - \varepsilon}). \]

This is true without any hypothesis. For if the Riemann hypothesis is false, Theorem 14.25 (C) shows that
\[ M(x) = O(x^{1 - \varepsilon}). \]

\( \dagger \) Landau (13), Diahann (1).
with some $a$ greater than $\frac{1}{2}$. On the other hand, if the Riemann hypothesis is true, then for $\sigma > \frac{1}{2}$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \mathcal{M}(n) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = s \int \frac{\mathcal{M}(s)}{s^s} \, ds.$$  

Suppose that

$$|\mathcal{M}(n)| \leq M_4 \quad (1 \leq n < a_3), \quad \leq \delta n^\sigma \quad (n \geq a_3).$$

Then

$$\left| \frac{1}{\zeta(s)} \right| \leq |\mu| M_4 \int_1^{a_3} \left| \frac{d}{dx} \left( \frac{1}{x^s} \right) \right| \, dx + \int_{a_3}^{\infty} \left| \frac{d}{dx} \left( \frac{\mu(x)}{x^s} \right) \right| \, dx \leq 2 |\mu| M_4 \int_1^{a_3} \frac{d}{dx} \left( \frac{1}{x^s} \right) \, dx + \frac{|\mu|}{s-\sigma}.$$  

(14.26.4)

But if $\rho = \frac{1}{2} + it$ is a simple zero of $\zeta(s)$, and $s = \sigma + it$, $\sigma > \frac{1}{2}$, then

$$\frac{1}{\zeta(s)} = \frac{1}{\zeta(\rho)} \sim \frac{1}{\rho}.$$  

We therefore obtain a contradiction if

$$\delta < \frac{1}{\rho}.$$  

This proves the theorem.

14.27. Formulas connecting the functions of prime-number theory with series of the form

$$\sum_{n=1}^{\infty} \frac{\rho_{p,n}}{n^s}.$$  

etc., are well known, and are discussed in the books of Landau and Ingham. Here we prove a similar formulas for the function $\mathcal{M}(s)$.

**Theorem 14.27.** There is a sequence $T_n$, $n \leq T_n \leq n + 1$, such that

$$\mathcal{M}(s) = \lim_{n \to \infty} \sum_{n \leq T_n} \rho_{p,n} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right).$$  

(14.27.1)

If $x$ is not an integer. If $x$ is an integer, $\mathcal{M}(x)$ is to be replaced by $\mathcal{M}(x-1)$.

In writing the series we have supposed for simplicity that all the zeros of $\zeta(s)$ are simple; obvious modifications are required if this is not so.

For a fixed non-integral $x$, (3.12.1), with $a_n = \rho(s)$, $s \to 0$, $c = 2$, and is replaced by $s$, gives

$$\mathcal{M}(s) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{T-i\infty}^{T+i\infty} \frac{1}{\zeta(s)} \, ds + O(1).$$  

If $x$ is an integer, $\frac{1}{\zeta(x)}$ is to be subtracted from the left-hand side. By the calculus of residues, the first term on the right is equal to

$$\sum_{n=T}^{\infty} \left( \frac{1}{n^{s-1}} - \frac{1}{n^{s+1}} \right) \frac{\mathcal{M}(n)}{n^s} + \frac{1}{2\pi i} \left( \int_{T-i\infty}^{T+i\infty} + \sum_{n=T}^{\infty} \int_{T-i\infty}^{T+i\infty} \frac{1}{n^{s-1}} \, ds \right),$$  

where $T$ is not the ordinate of a zero. Now

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mathcal{M}(n)}{n^s} = \frac{1}{s-1} \int_{T-i\infty}^{T+i\infty} \frac{1}{s-1} \frac{1}{\zeta(s)} \, ds.$$  

Here

$$\frac{1}{\zeta(s)} - O(s^{-\delta}) = O(s^{-\delta}) = O(s^{-\delta}) = O(s^{-\delta}).$$  

Hence the integral is

$$O\left( \int_{T-i\infty}^{T+i\infty} \frac{1}{s^x} \, ds \right),$$  

which tends to zero as $N \to \infty$, for a fixed $T$. Hence we obtain

$$\sum_{T \leq \rho_{p,n} \leq N} \frac{\rho_{p,n}}{n^s} = \sum_{n=T}^{\infty} \left( \frac{1}{n^{s-1}} - \frac{1}{n^{s+1}} \right) \frac{1}{2\pi i} \left( \int_{T-i\infty}^{T+i\infty} + \sum_{n=T}^{\infty} \int_{T-i\infty}^{T+i\infty} \frac{1}{n^{s-1}} \, ds \right).$$  

Also

$$\int_{T}^{\infty} \frac{\rho_{p,n}}{n^s} \, ds = \int_{T}^{\infty} \frac{1}{n^{s-1}} \, ds - \int_{T}^{\infty} \frac{1}{n^{s-1}} \cos \left( \pi s \right) \, ds \approx O \left( \frac{1}{\rho} \right).$$  

(14.27)
Also by (14.16.2) we can choose \( T = T_n \) such that 
\[
\frac{1}{\zeta(s)} = O(r) \quad (1 \leq \sigma \leq 2, \ t = T_n).
\]
Hence for \(-1 \leq \sigma \leq \frac{1}{2}, \ t = T_n\),
\[
\frac{1}{\zeta(s)} = O\left(\frac{|\psi(s) - \frac{1}{2s}|}{\zeta(1-\sigma)}\right) = O(r).
\]
Hence
\[
\sum_{1 \lt \delta \leq T_n} \frac{1}{\zeta(\delta)} ds - O(T^{1+\epsilon}).
\]
Similarly for the integral over \((2-\sigmaT, -\infty-\sigmaT)\), and the result stated follows.

It follows from the above theorem that
\[
\sum_{\delta > 0} \frac{1}{\zeta(\delta)}
\]
is divergent; if it were convergent,
\[
\sum_{\delta > 0} \frac{1}{\zeta(\delta)}
\]
would be uniformly convergent over any finite interval, and \( M(x) \)
would be continuous.

14.28. The Mertens hypothesis.† It was conjectured by Mertens,
from numerical evidence, that
\[
|M(x)| \leq \frac{x}{\log x} \quad (x \geq 1). \tag{14.28.1}
\]
This has not been proved or disproved. It implies the Riemann hypothesis,
but is not apparently a consequence of it. A slightly less precise
hypothesis would be
\[
M(x) = O(x). \tag{14.28.2}
\]
The problem has a certain similarity to that of the function \( \psi(x) - x \)
in prime-number theory, where
\[
\psi(x) = \sum_{n \leq x} \Lambda(n).
\]
On the Riemann hypothesis, \( \psi(x) - x = O(x^{1/2+\epsilon}) \), but it is not of the
form \( O(x^{1/2}) \), and in fact
\[
d(x) - x = O(x^{1/2} \log \log x). \tag{14.28.3}
\]
The influence of the factor \( \log \log x \) is quite inappreciable as far as
\[
\text{the calculations go, and it might be conjectured that (14.28.2) could be}
\]
disproved similarly. We shall show, however, that there is an essential
difference between the two problems, and that the proof of (14.28.3)
cannot be extended to the other case, at any rate in any obvious way.

The proof of (14.28.3) depends on the fact that the real part of
\[
\sum_{\delta > 0} \frac{1}{\zeta(\delta)}
\]
is unbounded in the neighbourhood of \( z = 0 \). To deal with \( M(x) \)
in the same way, we should have to prove that the real part of
\[
f(x) = \sum_{\delta > 0} \frac{1}{\zeta(\delta)} \psi'(\delta z) \tag{14.29.1}
\]
is unbounded in the neighbourhood of \( z = 0 \). This, however, is not the
case. For consider the integral
\[
\frac{1}{2\pi i} \int_{\frac{1}{2}-it}^{\frac{1}{2}+it} \frac{z^{s}}{\zeta(s)} ds
\]
taken round the rectangle \((-1, 2, -i\infty, i\infty, -1+i\infty)\), where the \( T_n \)
are those of the previous section, and an indentation is made above \( z = 0 \).

The integral along the upper side of the contour tends to 0 as \( n \to \infty \),
and we calculate that
\[
f(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-it}^{\frac{1}{2}+it} \frac{z^{s}}{\zeta(s)} ds - \frac{1}{2\pi i} \int_{1/2}^{1/2+it} \frac{z^{s}}{\zeta(s)} ds + \frac{1}{2\pi i} \int_{1/2}^{1/2} \frac{z^{s}}{\zeta'(s)} ds.
\]
The last term tends to a finite limit as \( x \to 0 \). Also
\[
\int e^{-x} - e^{-\frac{x}{2}} \leq e^{-x} \quad (x = -1 + it, x = -1 + it, x > 0)
\]
and
\[
\int \frac{1}{\log x} = O(1). \tag{14.29.2}
\]
The second term is therefore bounded for \( R(s) > 0 \).

The first term is equal to
\[
\int_{1/2}^{1/2+it} \frac{1}{\log s} ds
\]
Now, if \( n > 1 \),
\[
\int_{1}^{2} \frac{1}{\log s} ds = \left[ \frac{1}{\log s} \right]_{1}^{2} + \int_{1}^{2} \frac{1}{\log s} ds = \int_{1}^{2} \frac{1}{\log s} ds
\]
and
\[
\int_{e^{\log n} - 1}^{e^{\log n}} \frac{1}{\log s} ds = e^{\log n} - e^{\log n} = n \log n.
\]
Hence
\[
\int_{1}^{2} \frac{1}{\log s} ds = O(1).
\]
uniformly in the neighbourhood of \( z = 0 \). Hence
\[
\sum_{n < x} \frac{1}{n} \int_{x}^{x+1} \frac{e^{i \alpha n}}{n} \, dx = O(1).
\]
If \( s = re^{i \alpha} \), we have
\[
\int_{x}^{x+1} \frac{e^{i \alpha n}}{n} \, dx = \int_{x}^{x+1} e^{i \alpha n} \frac{1}{n} \, dx \\
= O(1) + \int_{x}^{x+1} e^{i \alpha n} \frac{1}{n} \, dx \\
= O(1) + \int_{x}^{x+1} e^{i \alpha n} \frac{1}{n} \, dx \\
= O(1) + \int_{x}^{x+1} \frac{d e^{i \alpha n}}{e^{i \alpha n}} \, dx + \int_{x}^{x+1} e^{i \alpha n} \frac{1}{n} \, dx \\
= \log \frac{1}{r} + O(1).
\]
Hence
\[
f(s) = \frac{1}{2 \pi i} \log \frac{1}{r} + O(1),
\]
and consequently \( R(s) \) is bounded.

14.29. In this section we shall investigate the consequences of the hypothesis that
\[
\int_{1}^{X} \left( \frac{M(x)}{x} \right) \frac{1}{x} \, dx = O(\log X).
\]
This is less drastic than the Mertens hypothesis, since it clearly follows from (14.28.3). The corresponding formula with \( M(x) \) replaced by \( \zeta(x) \) is a consequence of the Riemann hypothesis.

**Theorem 14.29 (a).** If (14.29.1) is true, all the zeros of \( \zeta(x) \) on the critical line are simple.

By (14.28.3),
\[
\begin{align*}
\int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx &= \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&= \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&= \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&\leq \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&\leq \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&\leq \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx \\
&\leq \int_{1}^{X} \frac{M(x)}{x} \frac{1}{x} \, dx.
\end{align*}
\]

\[\text{Grande (5).}\]
In the first sum, the terms with \( \rho' = 1 - \rho \) are
\[
\sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \int \frac{dx}{\gamma + \rho} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1},
\]
since \( 1 - \rho \) is the conjugate of \( \rho \). In the remaining terms, \( \rho = \frac{1}{2} + i\gamma \).
\( \rho' = \frac{1}{2} + i\gamma' \), where \( \gamma' \neq -\gamma \). Hence
\[
\int \frac{x^\rho x^{-1}}{\rho + \rho' - 1} \, dx = \frac{X^{x_{\rho} - 1} - 1}{\rho + \rho' - 1} = O \left( \frac{1}{\gamma + \gamma'} \right).
\]
Hence the sum of these terms is less than \( K_1 = K_1(T) \).

In the last sum we write
\[
\int \frac{X^{x_{\rho} - 1}}{\rho + \rho' - 1} \, dx = \int \frac{1}{\rho + \rho'} \int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx + \frac{1}{\rho + \rho'} \int M(x) x^{\rho - 1} \, dx.
\]
The last term is
\[
O \left( \frac{1}{\rho + \rho'} \int \frac{1}{\rho + \rho'} \int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx \right) = O \left( \frac{1}{\rho + \rho'} \int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx \right) = O \left( \frac{1}{\rho + \rho'} \int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx \right) = O \left( \frac{1}{\rho + \rho'} \int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx \right)
\]
by (14.29.1). Also
\[
\int \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx = \frac{1}{2\pi i} \int_{x=1}^{x=+\infty} \frac{X^{x_{\rho} - 1}}{\rho + \rho'} \, dx.
\]
(14.29.5)

To prove theorems, the residues sum to \( \frac{1}{2\pi i} \int U(w) \, dw \) on the right-hand side and integrate term by term. This is justified by absolute convergence. We obtain
\[
\sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \int \frac{dx}{\gamma + \rho - 1} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \int
\]
Evaluating the integral in the usual way by the calculus of residues, we obtain
\[
\sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \int \frac{dx}{\gamma + \rho - 1} \sum_{\gamma \in \tau' \leq t} \frac{1}{\gamma + \rho - 1} \int
\]
and (14.29.5) follows.
By the functional equation the same result then holds for \( \frac{1}{2} < \sigma \leq \frac{1}{2} \) also. Hence

\[
X^{1-\sigma} \int_{1-\sigma}^{X} \frac{1}{\zeta(s)} \frac{x^{1-\sigma}}{x^{1-\sigma}} \, dx = O\left( \frac{X^{1}}{U (1 + \frac{T}{U})} \right) = O\left( \frac{X^{1}}{U (1 + \frac{T}{U})} \right)
\]

and similarly for the integral over \((2-iU, \frac{1}{2}-iU)\). Making \( U \to \infty \), it follows that

\[
\int_{\frac{1}{2}-iU}^{1-iU} M(s) \, x^{s-\frac{1}{2}} \, dx = \frac{\log X}{(1-\sigma)(1-\rho)} + R,
\]

where \(|R| < K = K_2(T)\) if \(|\gamma| < T\).

Hence we obtain

\[
0 \leq 4 \log X \log X \sum_{\rho \neq \frac{1}{2}} \frac{1}{|\rho|^{1-s}} \sum_{\beta \neq 1} \frac{1}{|\beta|^{1-s}} \sum_{\beta \neq 1} \frac{1}{|\beta|^{1-s}}
\]

\[
+ A \log \log X + K_4(T),
\]

\[
\sum_{\gamma \in \gamma} \frac{1}{|\gamma|^{1-s}} \leq A + \frac{A}{\log \log X}.
\]

Making \( X \to \infty \),

\[
\sum_{\gamma \in \gamma} \frac{1}{|\gamma|^{1-s}} \leq A.
\]

Since the right-hand side is now independent of \( T \), the result follows,

In particular

\[
\frac{1}{\zeta(s)} = o(1/R).
\]

14.30. If (14.29.1) is true,†

\[
C \left( \frac{1}{2} + it \right) = O(\log t).
\]

Suppose that the interval \((t-\varepsilon, t+\varepsilon)\) contains \( \gamma \), the ordinate of a zero. By differentiating (2.4) twice,

\[
C(\frac{1}{2} + it) = O(t).
\]

Using this and (14.29.3), we obtain

\[
C(\frac{1}{2} + it) = \frac{1}{\zeta(1 + it)} + \frac{1}{\zeta(1 - it)} \, dt
\]

\[
> \frac{A}{2} \log^{-1} \left( -t \right)
\]

\[
> \frac{A}{t} - \frac{A}{t} - \frac{A}{t}.
\]

† Grammel and Landau (1).
by (14.14.1) and the functional equation. Taking \( \sigma = 1/\log t \),

\[
\Gamma(1 + i) = O(\exp\left(\frac{\log^2 t}{\log(\log t)}\right)),
\]

and the result follows.

14.32. Necessary and sufficient conditions for the Riemann hypothesis.

Two such conditions have been given in § 14.25. Other similar conditions occur in the prime-number problem.\( \dagger \)

A different kind of condition was stated by M. Riesz.\( \ddagger \) Let

\[
F(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}s}{(n-1)! (2\pi)^n}.
\]

(14.32.1)

Then a simple application of the calculus of residues gives

\[
F(s) = \frac{s}{2\pi} \int_{-\infty}^{\infty} \frac{e^{2\pi i y s}}{\Gamma(1 + s) \sin \pi s} \, ds = \frac{s}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\Gamma(1 + s)} \, ds,
\]

where \( \frac{1}{2} < a < 1 \). Taking \( s \) just greater than \( \frac{1}{2} \), it clearly follows that

\[
F(s) = O(e^{\pi s}).
\]

On the Riemann hypothesis we could move the line of integration to \( s = \frac{1}{2} + \sigma \) (using (14.23.5)) and obtain similarly

\[
F(s) = O(e^{\pi s}).
\]

(14.32.2)

Conversely, by Mellin's inversion formula,

\[
\frac{1}{\zeta(s)} = \int_{-\infty}^{\infty} F(s)e^{-s\sigma} \, ds.
\]

If (14.32.2) holds, the integral converges uniformly for \( \sigma > \sigma_0 > \frac{1}{2} \); the analytic function represented is therefore regular for \( \sigma > \frac{1}{2} \), and the truth of the Riemann hypothesis follows. Hence (14.32.3) is a necessary and sufficient condition for the Riemann hypothesis.

A similar condition stated by Hardy and Littlewood\( \S \) is

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1+s}} = O(s),
\]

(14.32.3)

These conditions have a superficial attractiveness since they depend explicitly only on values taken by \( \zeta(s) \) at points in \( s > 1 \); but actually no use has ever been made of them.

\( \dagger \) Landau, Vorlesungen, ii, 106-114.

\( \ddagger \) Hardy and Littlewood (3).

\( \S \) M. Riesz (1).

14.32. Conditions for the Riemann hypothesis also occur in the theory of Jacobi series. Let the fractions \( h/k \) with \( 0 < h \leq k, (h, k) = 1, k \leq N \), arranged in order of magnitude, be denoted by \( r_i \) (\( i = 1, 2, \ldots, \Phi(N) \)), where \( \Phi(N) = \phi(1) + \cdots + \phi(N) \). Let

\[
\delta_i = r_i - \eta_i(\Phi(N))
\]

be the distance between \( r_i \) and the corresponding fraction obtained by dividing up the interval \((0, 1)\) into \( \Phi(N) \) equal parts. Then a necessary and sufficient condition for the Riemann hypothesis is:

\[
\sum_{1 \leq i \leq \phi(N)} \delta_i^2 = O\left(\frac{1}{N^{1+\epsilon}}\right).
\]

(14.32.4)

An alternative necessary and sufficient condition is:

\[
\sum_{1 \leq i \leq \phi(N)} |\delta_i| = O(N^{1+\epsilon}).
\]

(14.32.5)

Details are given in Landau's Vorlesungen, ii, 167-177.

Still another condition can be expressed in terms of the formula of § 10.1. If \( \Xi(t) \) and \( \Phi(s) \) are related by (10.1.3), a necessary and sufficient condition that all the zeros of \( \Xi(t) \) should be real is that

\[
\int_{-\infty}^{\infty} \Phi(s)\Phi(\beta) e^{i\pi \beta^2} e^{-\pi \beta^2} e^{-\pi \beta^2} \, d\beta \geq 0
\]

(14.32.6)

for all real values of \( x \) and \( y \). But no method has been suggested of showing whether such criteria are satisfied or not.

A sufficient condition \( \ddagger \) for the Riemann hypothesis is that the partial sums \( \sum_{n=1}^{N} n^{-s} \) of the series for \( \zeta(s) \) should have no zeros in \( \sigma > 1 \).

\( \ddagger \) Jensen (1).

\( \S \) Landau (16).

\( \S \) see Zygmund (2), § 7.

\( \ddagger \) Turan (3).
uniformly for \( a \leq \sigma < \frac{1}{2} \) and \( t \geq 2 \), and hence

\[
\log \xi(s) = \begin{cases} \log \frac{2}{\sigma - 1} & \text{if } 1 + \frac{1}{\log \log t} < c < \frac{1}{2}, \\ \frac{(\log t)^{-1} + \log \log t}{(1 - c) \log \log t} & \text{if } a \leq \sigma \leq 1 + \frac{1}{\log \log t}. \end{cases}
\]

These results, together with those of §14.14 are the sharpest conditional order-estimates available at present.

14.34. The \( \Omega \)-result given by Theorem 14.12(A) has been sharpened by Montgomery [3], to give

\[
S(t) = \Omega \left( \frac{\log \log t}{\log \log \log t} \right)
\]

on the Riemann hypothesis. A minor modification of his method also yields

\[
S(t) = \Omega \left( \frac{\log \log t}{\log \log \log t} \right).
\]

It may be conjectured that these are best possible.

Mueller [2] has shown, on the Riemann hypothesis, that if \( c \) is a suitable constant, then \( S(t) \) changes sign in any interval \( [T, T + c \log T] \).

Further results and conjectures on the vertical distribution of the zeros are given by Montgomery [3], who investigated the pair correlation function

\[
F(s, T) = \frac{1}{N(T)} \sum_{0 \leq a \leq x \leq T} T^{1-\sigma} \tau(y - y'),
\]

where \( \tau(s) = 4/(4 + \sigma) \). This is a real-valued, even, non-negative function of \( \sigma \), and satisfies

\[
F(s, T) = s + T - \log T + O\left( \frac{1}{\log T} \right) + O(T^{-1}) + O(T^{-1/2})
\]

for \( s > 0 \), whence \( F(s, T) \to s \) as \( T \to \infty \), uniformly for \( 0 < \delta \leq \sigma < 1 - \delta \). Montgomery conjectured that in general

\[
F(s, T) \to \min(a, 1)
\]

for \( 0 < a < \frac{1}{2} \) and \( t \geq 2 \), and hence

\[
\log \xi(s) = \begin{cases} \log \frac{2}{\sigma - 1} & \text{if } 1 + \frac{1}{\log \log t} < c < \frac{1}{2}, \\ \frac{(\log t)^{-1} + \log \log t}{(1 - c) \log \log t} & \text{if } a \leq \sigma \leq 1 + \frac{1}{\log \log t}. \end{cases}
\]

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\]

where \( \tau(s) = 4/(4 + \sigma) \). This is a real-valued, even, non-negative function of \( \sigma \), and satisfies

\[
F(s, T) = s + T - \log T + O\left( \frac{1}{\log T} \right) + O(T^{-1}) + O(T^{-1/2})
\]

for \( s > 0 \), whence \( F(s, T) \to s \) as \( T \to \infty \), uniformly for \( 0 < \delta \leq \sigma < 1 - \delta \). Montgomery conjectured that in general

\[
F(s, T) \to \min(a, 1)
\]
write the sums occurring in (14.34.3) as integrals
\[ \frac{1}{2\pi i} \int_{P} \frac{\zeta'(s)}{\zeta(s)} M(s) \zeta(1-s) \, ds, \]
and
\[ \frac{1}{2\pi i} \int_{P} \frac{\zeta'(s)}{\zeta(s)} M(1-s) \zeta(1-s) \, ds, \]
taken around an appropriate rectangular path $P$. The estimation of these is long and complicated, but leads ultimately to the lower bound
\[ N^a(T) \geq \frac{\theta^a}{\log T} N(T). \]
The estimate (14.34.4) has also been improved, firstly by Montgomery and Odlyzko [1], and then by Conrey, Ghosh and Gonek [1]. The latter work produces the constant 0.6732. The corresponding lower bound
\[ \limsup \frac{T_{2,1}}{\log T_{2,1}} \leq \lambda > 1 \] (14.34.6)
has been considered by Mueller [1], as well as in the two papers just cited. Here the best result known is that of Conrey, Ghosh, and Gonek [1], which has $\lambda = 2.337$. Indeed, further work by Conrey, Ghosh, and Gonek, which is in the course of publication at the time of writing, yields $\lambda = 2.68$ subject to the generalized Riemann hypothesis (i.e. a Riemann hypothesis for $\zeta(s)$ and all Dirichlet $L$-functions $L(s, \chi)$). Moreover it seems likely that this condition may be relaxed to the ordinary Riemann hypothesis with further work.

If one asks for bounds of the form (14.34.4) and (14.34.6) which are satisfied by a positive proportion of zeroes of (4.36.1) then one may take constants 0.77 and 1.32 (Conrey, Ghosh, Goldston, Gonek, and Heath-Brown [1]).

14.35. It should be remarked in connection with §14.24 that Selberg (4) proved Theorem 14.24 with error term $O(T)$, while the method here yields only $O(T(\log \log T))$. Moreover he obtained the error term $O(T(\log \log T)^{-1})$ for (14.24).

14.36. The argument of the final paragraph of §14.27 may be quantified, and then yields
\[ \sum_{\Re \sigma > 1} \frac{\zeta'(s)}{\zeta(s)} = 1 \rightarrow T, \]
uniformly for $T \rightarrow T_{0}$, assuming the Riemann hypothesis and that all the zeroes are simple. However a slightly better result comes from combining

14.36.4 CONSEQUENCES OF RIEMANN HYPOTHESIS

the asymptotic formula
\[ \sum_{\sigma > \sigma_{0}} \frac{1}{\zeta(1+it)} \sim \frac{1}{\tau} N(T \log T)^{2}, \]
of Gonek (2) with the bound (14.34.3). Using Hölder's inequality one may then derive the estimate
\[ \sum_{\tau_{0} < t < \tau} \frac{1}{\zeta(1+it)} \geq T, \]
where $\Sigma$ counts simple zero only, and $c > 0$ is a suitable numerical constant.

14.37. The Mertens hypothesis has been disproved by Odlyzko and te Riele [1], who showed that
\[ \limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} > 1.96 \]
and
\[ \liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} < -1.009. \]
Their treatment is indirect, and produces no specific $x$ for which $|M(x)| > x$. The method used is computational, and depends on solving numerically the inequalities occurring in Kronecker's theorem, so as to make the first few terms of (14.27.2) pull in the same direction. To this extent Odlyzko and te Riele follow the earlier work of Jankstai and Peyerimhoff [1], but they use a much more efficient algorithm for solving the Diophantine approximation problem.

14.38. Turán (3) conjectured that
\[ \sum_{n \leq x} \frac{\lambda(n)}{n} \geq 0 \] (14.38.1)
for all $x > 0$, where $\lambda(n)$ is the Liouville function, given by (1.2.14). He showed that his condition, given in §14.32, implies the above conjecture, which in turn implies the Riemann hypothesis. However Haselgrove (2) proved that (14.38.1) is false in general, thereby showing that Turán's condition does not hold. Later Spera (1) found that calculation that
\[ \sum_{n=1}^{N} \frac{\lambda(n)}{n} \]
has a zero in the region $\sigma > 1$. 

XV

CALCULATIONS RELATING TO THE ZEROS

15.1. It is possible to verify by means of calculation that all the complex zeros of \( \zeta(s) \) up to a certain point lie exactly (not merely approximately) on the critical line. As a simple example we shall find roughly the position of the first complex zero in the upper half-plane, and show that it lies on the critical line.

We consider the function \( Z(t) = e^{\theta t}(1+it) \) defined in § 4.17. This is real for real values of \( t \), so that, if \( Z(t) \) and \( Z(t_0) \) have opposite signs, \( Z(t) \) vanishes between \( t \) and \( t_0 \), and so \( \zeta(s) \) has a zero on the critical line between \( \frac{1}{2} + it \) and \( \frac{1}{2} + it_0 \).

It follows from (2.2.1) that \( \zeta(1) < 0 \), then from (2.1.12) that \( \zeta(1) > 0 \), i.e. that \( Z(0) < 0 \), and then from (4.17.3) that \( Z(0) < 0 \).

We shall next consider the value \( t = 6\pi \). Now the argument of § 4.14 shows that, if \( x \) is half an odd integer,

\[
|\zeta(\frac{1}{2} + ix)| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+ix}} \lesssim \frac{s^{\frac{1}{2}+ix}}{\Gamma(1-s)} + \frac{2s^{1/2}}{1-s}
\]

(15.1.1)

Hence, taking \( t > 0 \),

\[
|Z(\frac{1}{2} + ix)| = \sum_{n=1}^{\infty} \frac{\cos(t \log n - \theta_n)}{n^{\frac{1}{2}} - i \frac{1}{2} - 1} \lesssim \frac{x^{1/2}}{t} + \frac{2x}{2\pi - t}
\]

(15.1.2)

For \( x = \frac{1}{2} \), \( t = 6\pi \), the right-hand side is about 0.6.

We next require an approximation to \( \theta \). We have

\[
e^{-\theta} \approx \frac{1 + it}{1 + it_0} = e^{i(t - t_0)}
\]

so that

\[
\theta \approx -i\log t + i\log (1 + it_0) - i\log \frac{1 - it}{1 - it_0}
\]

It may be verified that the term \( O(1/t) \) is negligible in the calculations.

Writing \( \theta = 2\pi K \), and using the values

\[
\log 2 = 0.6931, \quad \log 3 = 1.0986
\]

it is found that

\[
K \approx 0.1166, \quad 3\log 3 - K = 3.139\text{,}3\log 4 - K = 4.043
\]

approximately. Hence the cosines in (15.1.2) are all positive, and \( \cos 2\pi K = 0.74\ldots \). Hence \( Z(6\pi) > 0 \).

There is therefore one zero at least on the critical line between \( t = 0 \) and \( t = 6\pi \).

Again, the formula of § 9.3 gives

\[
N(T) = 1 + 2K + \frac{1}{\pi} \Delta \arg \zeta(\sigma)
\]

where \( \Delta \) denotes variation along \((2, 2+1T, \frac{1}{2}+1T)\). Now \( R(\pi) > 0 \) on \( \sigma = 2 \), and an argument similar to that already used, but depending on (15.1.1), shows that \( R(\pi) > 0 \) on \((2+1T, \frac{1}{2}+1T)\), if \( T = 6\pi \). Hence \( |\Delta \arg \zeta(\sigma)| < \frac{\pi}{2} \), and

\[
N(6\pi) < 1 + 2K < 2
\]

Hence there is at most one complex zero with imaginary part less than \( 6\pi \), and in fact just one, namely the one on the critical line.

15.2. It is plain that the above process can be continued as long as the appropriate changes of sign of the function \( Z(t) \) occur. Defining \( K = \frac{1}{2} \), as before, let \( t_{\nu} \) be such that

\[
\zeta(t_{\nu}) = -e - 1 \quad (\nu = 1, 2, \ldots)
\]

Then (15.1.2) gives

\[
Z(t_{\nu}) \sim (-1)^{\nu} \sum_{n=1}^{\infty} \frac{\cos(n \log n)}{n^\frac{1}{2}}
\]

If the sum is dominated by its first term, it is positive, and so \( Z(t_{\nu}) \) has the sign of \(-1)^{\nu} \). If this is true for \( \nu \) and \( \nu + 1 \), \( Z(t_{\nu}) \) has a zero in the interval \((t_{\nu}, t_{\nu + 1})\).

The value \( t = 6\pi \) in the above argument is a rough approximation to \( t_{\nu} \).

The ordinates of the first six zeros are

\[
14.13, 21.02, 22.01, 39.42, 32.93, 37.58
\]

to two decimal places.† Some of these have been calculated with great accuracy.

15.3. The calculation which the above process requires are very laborious if \( 6\pi \) is at all large. A much better method is to use the formula (4.17.5) arising from the approximate functional equation. Let us write \( t = 2\pi s \),

\[
y(s) = \frac{1}{\Gamma(1+s)} \cos(s \log n)
\]

and

\[
\Lambda(s) = \frac{\cos(2\pi(2 - s - 1/2y))}{\cos 2\pi y}
\]

† See the references Gram (6), Landau (7), in Landau's Handbuch.
Then (4.17.5) gives
\[ Z(2n\pi) = \sum_{n=1}^{\infty} \alpha_n(u) + (-1)^{n-1}u^{-\frac{1}{2}}(1 - \frac{1}{n}) R(u), \]
where \( m = \lfloor u \rfloor \), and \( R(u) = O(u^{-\frac{1}{2}}) \). The \( \alpha_n(u) \) can be found, for given values of \( u \), from a table of the function \( \cos 2\pi x \). In the interval \( 0 \leq x < 1 \), \( h(x) \) decreases steadily from 0.92288 to 0.92286, and \( h(1 - e^{-1}) = h(0) \).

For the purpose of calculation we require a numerical upper bound for \( R(u) \). A rather complicated formula of this kind is obtained in Titchmarsh (17), Theorem 2. For values of \( u \) which are not too small it can be much simplified, and in fact it is easy to deduce that
\[ \| R(u) \| < \frac{3}{2\pi^2} (u > 120). \]

This inequality is sufficient for most purposes.

Occasionally, when \( Z(2n\pi) \) is too small, a second term of the Riemann-Siegel asymptotic formula has to be used.

The values of \( u \) for which the calculations are performed are the solutions of (15.2.1), since they make \( \alpha_n \) alternately 1 and \(-1\). In the calculations described in Titchmarsh (17), I began with
\[ u = 1.6, \quad K = \frac{199}{2}, \quad \text{and went as far as} \quad u = 62,785, \quad K = 98,501. \]

The values of \( u \) were obtained in succession, and are rather rough approximations to the \( \alpha_n \), so that the \( K' \)'s are not quite integers or integers and a half.

It was shown in this way that the first 198 zeros of \( \zeta(s) \) above the real axis all lie on the line \( \sigma = \frac{1}{2} \).

The calculations were carried a great deal farther by Dr. Comrie. Proceeding on the same lines, it was shown that the first 1,041 zeros of \( \zeta(s) \) above the real axis all lie on the critical line, in the interval \( 0 < t < 1,400 \).

One interesting point which emerges from these calculations is that \( Z(0) \) does not always have the same sign as \(-1\). A considerable number of exceptional cases were found; but in each of these cases there is a neighbouring point \( \zeta \), such that \( Z(\zeta) \) has the sign of \(-1\), and the succession of changes of sign of \( Z(t) \) is therefore not interrupted.

15.4. As far as they go, these calculations are all in favour of the truth of the Riemann hypothesis. Nevertheless, it may be that they do not go far enough to reveal the real state of affairs. At the end of the table constructed by Dr. Comrie there are only fifteen terms in the series for \( Z(t) \), and this is a very small number when we are dealing with oscillating series of this kind. Indeed there is one feature of the table which may suggest a change in its character farther on. In the main, the result is dominated by the first term \( \alpha_n \) and later terms more or less cancel out. Occasionally (e.g. at \( K = 450 \)) all, or nearly all, the numbers \( \alpha_n \) have the same sign, and \( Z(t) \) has a large maximum or minimum. As we pass from this to neighbouring values of \( t \), the first few \( \alpha_n \) undergo violent changes, while the later ones vary comparatively slowly. The term \( \alpha_n \) appears when \( u \to n^2 \), and here
\[ \cos 2\pi(t - n\log n) = \cos 2\pi(\log(u/n^2) - n - \frac{1}{2} + ...) \]
\[ = \cos 2\pi(\log u/n^2) + ... = (-1)^n \cos 2\pi + ..., \]

and
\[ \frac{d}{dt}(K - u \log k) = -\frac{1}{2} \log u - \log n - \frac{1}{192\pi^2} \log u^2 - ... \approx -\frac{1}{192\pi^2}. \]

At its first appearance in the table \( \alpha_n \) will therefore be approximately \((-1)^n \pi^2 \cos 2\pi + ..., \) and it will vary slowly for some time after its appearance.

It is conceivable that if \( t \), and so the number of terms, were large enough, there might be places where the smaller slowly varying terms would combine to overpower the few quickly varying ones, and so prevent the graph of \( Z(t) \) from crossing the zero line between successive maxima. There are too few terms in the table already constructed to test this possibility.

There are, of course, relations between the numbers \( \alpha_n \), which destroy any too simple argument of this kind. If the Riemann hypothesis is true, there must be some relation, at present hidden, which prevents the suggested possibility from ever occurring at all.

No doubt the whole matter will soon be put to the test of modern methods of calculation. Naturally the Riemann hypothesis cannot be proved by calculation, but, if it is false, it could be disproved by the discovery of exceptions in this way.

NOTES FOR CHAPTER 15

15.3. A number of workers have checked the Riemann hypothesis over increasingly large ranges. At the time of writing the most extensive calculation is that of van de Lune and te Riele (as reported in Odlyzko and te Riele (1)), who have found that the first \( 1 \times 10^9 \) non-trivial zeros are simple and lie on the critical line.
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