

QUANTUM FIELD THEORY ON CURVED BACKGROUNDS. II. SPACETIME SYMMETRIES

ARTHUR JAFFE AND GORDON RITTER

ABSTRACT. We study space-time symmetries in scalar quantum field theory (including interacting theories) on static space-times. We first consider Euclidean quantum field theory on a static Riemannian manifold, and show that the isometry group is generated by one-parameter subgroups which have either self-adjoint or unitary quantizations. We analytically continue the self-adjoint semigroups to one-parameter unitary groups, and thus construct a unitary representation of the isometry group of the associated Lorentzian manifold. The method is illustrated for the example of hyperbolic space, whose Lorentzian continuation is Anti-de Sitter space.

1. INTRODUCTION

The extension of quantum field theory to curved space-times has led to the discovery of many qualitatively new phenomena which do not occur in the simpler theory on Minkowski space, such as Hawking radiation; for background and historical references, see [2, 6, 18].

The reconstruction of quantum field theory on a Lorentz-signature space-time from the corresponding Euclidean quantum field theory makes use of Osterwalder-Schrader (OS) positivity [15, 16] and analytic continuation. On a curved background, there may be no proper definition of time-translation and no Hamiltonian; thus, the mathematical framework of Euclidean quantum field theory may break down. However, on static space-times there is a Hamiltonian and it makes sense to define Euclidean QFT. This approach was recently taken by the authors [11], in which the fundamental properties of Osterwalder-Schrader quantization and some of the fundamental estimates of constructive quantum field theory¹ were generalized to static space-times.

The previous work [11], however, did not address the analytic continuation which leads from a Euclidean theory to a real-time theory. In the present article, we initiate a treatment of the analytic continuation by constructing unitary operators which form a representation of the isometry group of the Lorentz-signature space-time associated to a static Riemannian space-time. Our approach is similar in spirit to that of Fröhlich [4] and of Klein and

Date: February 22, 2007.

¹For background on constructive field theory in flat space-times, see [8, 9].

Landau [13], who showed how to go from the Euclidean group to the Poincaré group without using the field operators on flat space-time.

This work also has applications to representation theory, as it provides a natural (functorial) quantization procedure which constructs nontrivial unitary representations of those Lie groups which arise as isometry groups of static, Lorentz-signature space-times. These groups are typically non-compact. For example, when applied to AdS_{d+1} , our procedure gives a unitary representation of the identity component of $SO(d, 2)$. Moreover, our procedure makes use of the Cartan decomposition, a standard tool in representation theory.

2. CLASSICAL SPACE-TIME

2.1. Structure of Static Space-Times.

Definition 2.1. A **quantizable static space-time** is a complete, connected orientable Riemannian manifold (M, g_{ab}) with a globally-defined (smooth) Killing field ξ which is orthogonal to a codimension-one hypersurface $\Sigma \subset M$, such that the orbits of ξ are complete and each orbit intersects Σ exactly once.

Throughout this paper, we assume that M is a quantizable static space-time. Definition 2.1 implies that there is a global time function t defined up to a constant by the requirement that $\xi = \partial/\partial t$. Thus M is foliated by time-slices M_t , and

$$M = \Omega_- \cup \Sigma \cup \Omega_+$$

where the unions are disjoint, $\Sigma = M_0$, and Ω_{\pm} are open sets corresponding to $t > 0$ and $t < 0$ respectively. We infer existence of an isometry θ which reverses the sign of t ,

$$\theta : \Omega_{\pm} \rightarrow \Omega_{\mp} \text{ such that } \theta^2 = 1, \theta|_{\Sigma} = \text{id}.$$

Fix a self-adjoint extension of the Laplacian, and let $C = (-\Delta + m^2)^{-1}$ be the resolvent of the Laplacian (also called the *free covariance*), where $m^2 > 0$. Then C is a bounded self-adjoint operator on $L^2(M)$. For each $s \in \mathbb{R}$, the Sobolev space $\mathcal{H}_s(M)$ is a real Hilbert space, defined as completion of $C_c^\infty(M)$ in the norm

$$(2.1) \quad \|f\|_s^2 = \langle f, C^{-s} f \rangle.$$

The inclusion $\mathcal{H}_s \hookrightarrow \mathcal{H}_{s+k}$ for $k > 0$ is Hilbert-Schmidt. Define $\mathcal{S} := \bigcap_{s < 0} \mathcal{H}_s(M)$ and $\mathcal{S}' := \bigcup_{s > 0} \mathcal{H}_s(M)$. Then

$$\mathcal{S} \subset \mathcal{H}_{-1}(M) \subset \mathcal{S}'$$

form a Gelfand triple, and \mathcal{S} is a nuclear space.

Recall that \mathcal{S}' has a natural σ -algebra of measurable sets (see for instance [7, 8, 17]). There is a unique Gaussian probability measure μ with mean zero and covariance C defined on the cylinder sets in \mathcal{S}' (see [7]).

More generally, one may consider a non-Gaussian, countably-additive measure μ on \mathcal{S}' and the space

$$\mathcal{E} := L^2(\mathcal{S}', \mu).$$

We are interested in the case that the monomials of the form $A(\Phi) = \Phi(f_1) \dots \Phi(f_n)$ for $f_i \in \mathcal{S}$ are all elements of \mathcal{E} , and for which their span is dense in \mathcal{E} . This is of course true if μ is the Gaussian measure with covariance C .

For an open set $\Omega \subset M$, let \mathcal{E}_Ω denote the closure in \mathcal{E} of the set of monomials $A(\Phi) = \prod_i \Phi(f_i)$ where $\text{supp}(f_i) \subset \Omega$ for all i . Of particular importance for Euclidean quantum field theory is the positive-time subspace

$$\mathcal{E}_+ := \mathcal{E}_{\Omega_+}.$$

2.2. The Operator Induced by an Isometry. Isometries of the underlying space-time manifold act on a Hilbert space of classical fields arising in the study of a classical field theory. For $f \in C^\infty(M)$ and $\psi : M \rightarrow M$ an isometry, define

$$f^\psi \equiv (\psi^{-1})^* f = f \circ \psi^{-1}.$$

Since $\det(d\psi) = 1$, the operation $f \rightarrow f^\psi$ extends to a bounded operator on $\mathcal{H}_{\pm 1}(M)$ or on $L^2(M)$. A treatment of isometries for static space-times appears in [11].

Definition 2.2. Let ψ be an isometry, and $A(\Phi) = \Phi(f_1) \dots \Phi(f_n) \in \mathcal{E}$ a monomial. Define the induced operator

$$(2.2) \quad \Gamma(\psi)A \equiv \Phi(f_1^\psi) \dots \Phi(f_n^\psi),$$

and extend $\Gamma(\psi)$ by linearity to the domain of polynomials in the fields, which is dense in \mathcal{E} .

3. OSTERWALDER-SCHRADER QUANTIZATION

3.1. Quantization of Vectors (The Hilbert Space \mathcal{H} of Quantum Theory). In this section we define the quantization map $\mathcal{E}_+ \rightarrow \mathcal{H}$, where \mathcal{H} is the Hilbert space of quantum theory. The existence of the quantization map relies on a condition known as Osterwalder-Schrader (or reflection) positivity. A probability measure μ on \mathcal{S}' is said to be *reflection positive* if

$$(3.1) \quad \int \overline{\Gamma(\theta)F} F d\mu \geq 0$$

for all F in the positive-time subspace $\mathcal{E}_+ \subset \mathcal{E}$. Let $\Theta = \Gamma(\theta)$ be the reflection on \mathcal{E} induced by θ . Define the sesquilinear form (A, B) on $\mathcal{E}_+ \times \mathcal{E}_+$ as $(A, B) = \langle \Theta A, B \rangle_{\mathcal{E}}$, so (3.1) states that $(F, F) \geq 0$.

Assumption 1 (O-S Positivity). *Any measure $d\mu$ that we consider is reflection positive with respect to the time-reflection Θ .*

Definition 3.1 (OS-Quantization). Given a reflection-positive measure $d\mu$, the Hilbert space \mathcal{H} of quantum theory is the completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the inner product given by the sesquilinear form (A, B) . Denote the quantization map Π for vectors $\mathcal{E}_+ \rightarrow \mathcal{H}$ by $\Pi(A) = \hat{A}$, and write

$$(3.2) \quad \langle \hat{A}, \hat{B} \rangle_{\mathcal{H}} = (A, B) = \langle \Theta A, B \rangle_{\mathcal{E}} \quad \text{for } A, B \in \mathcal{E}_+.$$

3.2. Quantization of Operators. The basic quantization theorem gives a sufficient condition to map a (possibly unbounded) linear operator T on \mathcal{E} to its quantization, a linear operator \hat{T} on \mathcal{H} . Consider a densely-defined operator T on \mathcal{E} , the unitary time-reflection Θ , and the adjoint $T^+ = \Theta T^* \Theta$. A preliminary version of the following was also given in [10].

Definition 3.2 (Quantization Condition I). The operator T satisfies QC-I if:

- i. The operator T has a domain $\mathcal{D}(T)$ dense in \mathcal{E} .
- ii. There is a subdomain $\mathcal{D}_0 \subset \mathcal{E}_+ \cap \mathcal{D}(T) \cap \mathcal{D}(T^+)$, for which $\hat{\mathcal{D}}_0 \subset \mathcal{H}$ is dense.
- iii. The transformations T and T^+ both map \mathcal{D}_0 into \mathcal{E}_+ .

Theorem 3.3 (Quantization I). *If T satisfies QC-I, then*

- i. *The operators $T|_{\mathcal{D}_0}$ and $T^+|_{\mathcal{D}_0}$ have quantizations \hat{T} and \hat{T}^+ with domain $\hat{\mathcal{D}}_0$.*
- ii. *The operators $\hat{T}^* = (\hat{T}|_{\hat{\mathcal{D}}_0})^*$ and \hat{T}^+ agree on $\hat{\mathcal{D}}_0$.*
- iii. *The operator $\hat{T}|_{\hat{\mathcal{D}}_0}$ has a closure, namely \hat{T}^{**} .*

Proof. We wish to define the quantization \hat{T} with the putative domain $\hat{\mathcal{D}}_0$ by

$$(3.3) \quad \hat{T}\hat{A} = \widehat{TA}.$$

For any vector $A \in \mathcal{D}_0$ and for any $B \in (\mathcal{D}_0 \cap \mathcal{N})$, it is the case that $\hat{A} = \widehat{A+B}$. The transformation \hat{T} is defined by (3.3) iff $\widehat{TA} = T(\widehat{A+B}) = \widehat{TA} + \widehat{TB}$. Hence one needs to verify that $T : \mathcal{D}_0 \cap \mathcal{N} \rightarrow \mathcal{N}$, which we now do.

The assumption $\mathcal{D}_0 \subset \mathcal{D}(T^+)$, along with the fact that Θ is unitary, ensures that $\Theta\mathcal{D}_0 \subset \mathcal{D}(T^*)$. Therefore for any $F \in \mathcal{D}_0$,

$$(3.4) \quad \langle \Theta F, TB \rangle_{\mathcal{E}} = \langle T^* \Theta F, B \rangle_{\mathcal{E}} = \langle \Theta (\Theta T^* \Theta F), B \rangle_{\mathcal{E}} = \langle \Theta T^+ F, B \rangle_{\mathcal{E}} = \langle \widehat{T^+ F}, \hat{B} \rangle_{\mathcal{H}}.$$

In the last step we use the fact assumed in part (iii) of QC-I that $T^+ : \mathcal{D}_0 \rightarrow \mathcal{E}_+$, yielding the inner product of two vectors in \mathcal{H} . We infer from the Schwarz inequality in \mathcal{H} that

$$|\langle \Theta F, TB \rangle_{\mathcal{E}}| \leq \|\widehat{T^+ F}\|_{\mathcal{H}} \|\hat{B}\|_{\mathcal{H}} = 0.$$

As $\langle \Theta F, TB \rangle_{\mathcal{E}} = \langle \hat{F}, \widehat{TB} \rangle_{\mathcal{H}}$, this means that $\widehat{TB} \perp \hat{\mathcal{D}}_0$. As $\hat{\mathcal{D}}_0$ is dense in \mathcal{H} by QC-I.ii, we infer $\widehat{TB} = 0$. In other words, $TB \in \mathcal{N}$ as required to define \hat{T} .

In order show that $\hat{\mathcal{D}}_0 \subset \mathcal{D}(\hat{T}^*)$, perform a similar calculation to (3.4) with arbitrary $A \in \mathcal{D}_0$ replacing B , namely

$$(3.5) \quad \langle \hat{F}, \hat{T}\hat{A} \rangle_{\mathcal{H}} = \langle \Theta F, TA \rangle_{\mathcal{E}} = \langle \Theta(\Theta T^* \Theta F), A \rangle_{\mathcal{E}} = \langle \Theta T^+ F, A \rangle_{\mathcal{E}} = \langle \widehat{T^+ F}, \hat{A} \rangle_{\mathcal{H}} .$$

The right side is continuous in $\hat{A} \in \mathcal{H}$, and therefore $\hat{F} \in \mathcal{D}(T^*)$. Furthermore $T^* \hat{F} = \widehat{T^+ F}$. This identity shows that if $F \in \mathcal{N}$, then $\widehat{T^+ F} = 0$. Hence $T^+ \upharpoonright \mathcal{D}_0$ has a quantization $\widehat{T^+}$, and we may write (3.5) as

$$(3.6) \quad T^* \hat{F} = \widehat{T^+ F} , \quad \text{for all } F \in \mathcal{D}_0 .$$

In particular \hat{T}^* is densely defined so \hat{T} has a closure. This completes the proof. \square

Definition 3.4 (Quantization Condition II). The operator T satisfies QC-II if

- i. Both the operator T and its adjoint T^* have dense domains $\mathcal{D}(T), \mathcal{D}(T^*) \subset \mathcal{E}$.
- ii. There is a domain $\mathcal{D}_0 \subset \mathcal{E}_+$ in the common domain of T, T^+, T^+T , and TT^+ .
- iii. Each operator T, T^+, T^+T , and TT^+ maps \mathcal{D}_0 into \mathcal{E}_+ .

Theorem 3.5 (Quantization II). *If T satisfies QC-II, then*

- i. *The operators $T \upharpoonright \mathcal{D}_0$ and $T^+ \upharpoonright \mathcal{D}_0$ have quantizations \hat{T} and $\widehat{T^+}$ with domain $\hat{\mathcal{D}}_0$.*
- ii. *If $A, B \in \mathcal{D}_0$, one has $\langle \hat{B}, \hat{T}\hat{A} \rangle_{\mathcal{H}} = \langle \widehat{T^+ B}, \hat{A} \rangle_{\mathcal{H}}$.*

Remarks.

- i. In Theorem 3.5 we drop the assumption that the domain $\hat{\mathcal{D}}_0$ is dense, obtaining quantizations \hat{T} and $\widehat{T^+}$ whose domains are not necessarily dense. In order to compensate for this, we assume more properties concerning the domain and the range of T^+ on \mathcal{E} .
- ii. As $\hat{\mathcal{D}}_0$ need not be dense in \mathcal{H} , the adjoint of \hat{T} need not be defined. Nevertheless, one calls the operator \hat{T} *symmetric* in case one has

$$(3.7) \quad \langle \hat{B}, \hat{T}\hat{A} \rangle_{\mathcal{H}} = \langle \hat{T}\hat{B}, \hat{A} \rangle_{\mathcal{H}} , \quad \text{for all } A, B \in \mathcal{D}_0 .$$

- iii. If $\hat{S} \supset \hat{T}$ is a densely-defined extension of \hat{T} , then $\hat{S}^* = \widehat{T^+}$ on the domain $\hat{\mathcal{D}}_0$.

Proof. We define the quantization \hat{T} with the putative domain $\hat{\mathcal{D}}_0$. As in the proof of Theorem 3.3, this quantization \hat{T} is well-defined iff it is the case that $T : \mathcal{D}_0 \cap \mathcal{N} \rightarrow \mathcal{N}$. For any $F \in \mathcal{D}_0 \cap \mathcal{N}$, by definition $\|\hat{F}\|_{\mathcal{H}} = 0$. Also

$$\langle TF, TF \rangle_{\mathcal{H}} = (TF, TF) = \langle \Theta TF, TF \rangle_{\mathcal{E}} = \langle F, T^* \Theta TF \rangle_{\mathcal{E}} ,$$

where one uses the fact that $\mathcal{D}_0 \subset \mathcal{D}(T^+T)$. Thus

$$\langle TF, TF \rangle_{\mathcal{H}} = \langle \Theta F, T^+ TF \rangle_{\mathcal{E}} = \langle F, T^+ TF \rangle_{\mathcal{H}} .$$

Here we use the fact that T^+T maps \mathcal{D}_0 to \mathcal{E}_+ . Thus one can use the Schwarz inequality on \mathcal{H} to obtain

$$\langle TF, TF \rangle_{\mathcal{H}} \leq \|\hat{F}\|_{\mathcal{H}} \|\widehat{T^+TF}\|_{\mathcal{H}} = 0.$$

Hence $T : \mathcal{D}_0 \cap \mathcal{N} \rightarrow \mathcal{N}$, and T has a quantization \hat{T} with domain $\hat{\mathcal{D}}_0$.

In order to verify that $T^+ \upharpoonright \mathcal{D}_0$ has a quantization, one needs to show that $T^+ : \mathcal{D}_0 \cap \mathcal{N} \subset \mathcal{N}$. Repeat the argument above with T^+ replacing T . The assumption $TT^+ : \mathcal{D}_0 \rightarrow \mathcal{E}_+$ yields for $F \in \mathcal{D}_0 \cap \mathcal{N}$,

$$\langle T^+F, T^+F \rangle_{\mathcal{H}} = \langle T^*\Theta F, T^+F \rangle_{\mathcal{E}} = \langle \Theta F, TT^+F \rangle_{\mathcal{E}} = \langle \hat{F}, \widehat{TT^+F} \rangle_{\mathcal{H}}.$$

Use the Schwarz inequality in \mathcal{H} to obtain the desired result that

$$\langle T^+F, T^+F \rangle_{\mathcal{H}} \leq \|\hat{F}\|_{\mathcal{H}} \|\widehat{TT^+F}\|_{\mathcal{H}} = 0.$$

Hence T^+ has a quantization $\widehat{T^+}$ with domain $\hat{\mathcal{D}}_0$, and for $B \in \mathcal{D}_0$ one has $\widehat{T^+B} = \widehat{T^+}\hat{B}$. In order to establish (ii), assume that $A, B \in \mathcal{D}_0$. Then

$$\begin{aligned} \langle \hat{B}, \hat{T}\hat{A} \rangle_{\mathcal{H}} &= \langle \Theta B, TA \rangle_{\mathcal{E}} = \langle \Theta (\Theta T^* \Theta B), A \rangle_{\mathcal{E}} = \langle \Theta T^+ B, A \rangle_{\mathcal{E}} \\ (3.8) \quad &= \langle \widehat{T^+B}, \hat{A} \rangle_{\mathcal{H}} = \langle \widehat{T^+}\hat{B}, \hat{A} \rangle_{\mathcal{H}}. \end{aligned}$$

□

This completes the proof.

4. STRUCTURE AND REPRESENTATION OF THE LIE ALGEBRA OF KILLING FIELDS

For the remainder of this paper we assume the following, which is clearly true in the Gaussian case as the Laplacian commutes with the isometry group G . (A further explanation was given in [11].)

Assumption 2. *The isometry groups G that we consider leave the measure $d\mu$ invariant, in the sense that Γ , defined above, is a unitary representation of G on \mathcal{E} .*

4.1. The Representation of \mathfrak{g} on \mathcal{E} .

Lemma 4.1. *Let G_i be an analytic group with Lie algebra \mathfrak{g}_i ($i = 1, 2$), and let $\lambda : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a homomorphism. There cannot exist more than one analytic homomorphism $\pi : G_1 \rightarrow G_2$ for which $d\pi = \lambda$. If G_1 is simply connected then there is always one such π .*

Let $D = d/dt$ denote the canonical unit vector field on \mathbb{R} . Let G be a real Lie group with algebra \mathfrak{g} , and let $X \in \mathfrak{g}$. The map $tD \rightarrow tX$ ($t \in \mathbb{R}$) is a homomorphism of $\text{Lie}(\mathbb{R}) \rightarrow \mathfrak{g}$, so by the Lemma there is a unique analytic homomorphism $\xi_X : \mathbb{R} \rightarrow G$ such that $d\xi_X(D) = X$. Conversely, if η is an analytic homomorphism of $\mathbb{R} \rightarrow G$, and if we let $X = d\eta(D)$, it is obvious that $\eta = \xi_X$. Thus $X \mapsto \xi_X$ is a bijection of \mathfrak{g} onto the set of analytic homomorphisms $\mathbb{R} \rightarrow G$. The exponential map is defined by

$\exp(X) := \xi_X(1)$. For complex Lie groups, the same argument applies, replacing \mathbb{R} with \mathbb{C} throughout.

Since \mathfrak{g} is connected, so is $\exp(\mathfrak{g})$. Hence $\exp(\mathfrak{g}) \subseteq G^0$, where G^0 denotes the connected component of the identity in G . It need not be the case for a general Lie group that $\exp(\mathfrak{g}) = G^0$, but for a large class of examples (the so-called *exponential groups*) this does hold. For any Lie group, $\exp(\mathfrak{g})$ contains an open neighborhood of the identity, so the subgroup generated by $\exp(\mathfrak{g})$ always coincides with G^0 .

We will apply the above results with $G = \text{Iso}(M)$, the isometry group of M , and $\mathfrak{g} = \text{Lie}(G)$ the algebra of global Killing fields. Thus we have a bijective correspondence between Killing fields and 1-parameter groups of isometries. This correspondence has a geometric realization: the 1-parameter group of isometries

$$\phi_s = \xi_X(s) = \exp(sX)$$

corresponding to $X \in \mathfrak{g}$ is the flow generated by X .

Consider the two different 1-parameter groups of unitary operators:

- (1) the unitary group ϕ_s^* on $L^2(M)$, and
- (2) the unitary group $\Gamma(\phi_s)$ on \mathcal{E} .

Stone's theorem applies to both of these unitary groups to yield densely-defined self-adjoint operators on the respective Hilbert spaces.

In the first case, the relevant self-adjoint operator is simply an extension of $-iX$, viewed as a differential operator on $C_c^\infty(M)$. This is because for $f \in C_c^\infty(M)$ and $p \in M$, we have:

$$X_p f = (\mathcal{L}_X f)(p) = \frac{d}{ds} f(\phi_s(p))|_{s=0}.$$

Thus $-iX$ is a densely-defined symmetric operator on $L^2(M)$, and Stone's theorem implies that $-iX$ has self-adjoint extensions.

In the second case, the unitary group $\Gamma(\phi_s)$ on \mathcal{E} also has a self-adjoint generator $\Gamma(X)$, which can be calculated explicitly. By definition,

$$e^{-is\Gamma(X)} \left[\prod_{i=1}^n \Phi(f_i) \right] = \prod_{i=1}^n \Phi(f_i \circ \phi_{-s}).$$

Now replace $s \rightarrow -s$ and calculate $d/ds|_{s=0}$ applied to both sides of the last equation to see that

$$\Gamma(X) \left[\prod_{i=1}^n \Phi(f_i) \right] = \sum_{j=1}^n \Phi(f_1) \dots \Phi(-iX f_j) \Phi(f_{j+1}) \dots \Phi(f_n).$$

One may check that Γ is a Lie algebra representation of \mathfrak{g} , i.e. $\Gamma([X, Y]) = [\Gamma(X), \Gamma(Y)]$.

4.2. The Cartan Decomposition of \mathfrak{g} . For each $\xi \in \mathfrak{g}$, there exists some dense domain in \mathcal{E} on which $\Gamma(\xi)$ is self-adjoint, as discussed previously.

However, the quantizations $\widehat{\Gamma}(\xi)$ acting on \mathcal{H} may be hermitian, anti-hermitian, or neither depending on whether there holds a relation of the form

$$(4.1) \quad \Gamma(\xi)\Theta = \pm\Theta\Gamma(\xi),$$

with one of the two possible signs, or whether no such relation holds.

Even if (4.1) holds, to complete the construction of a unitary representation one must prove that there exists a dense domain in \mathcal{H} on which $\widehat{\Gamma}(\xi)$ is self-adjoint or skew-adjoint. This nontrivial problem will be dealt with in a later section using Theorems 3.3 and 3.5 and the theory of symmetric local semigroups [12, 4]. Presently we determine *which* elements within \mathfrak{g} satisfy relations of the form (4.1).

Let $\vartheta := \theta^*$ as an operator on $C^\infty(M)$, and consider a Killing field $X \in \mathfrak{g}$ also as an operator on $C^\infty(M)$. Define $\mathcal{T} : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$(4.2) \quad \mathcal{T}(X) := \vartheta X \vartheta.$$

From (4.2) it is not obvious that the range of \mathcal{T} is contained in \mathfrak{g} . To prove this, we recall some geometric constructions.

Let M, N be manifolds, let $\psi : M \rightarrow N$ be a diffeomorphism, and $X \in \text{Vect}(M)$. Then

$$(4.3) \quad \psi^{-1*} X \psi^* = X(\cdot \circ \psi) \circ \psi^{-1}.$$

defines an operator on $C^\infty(N)$. One may check that this operator is a derivation, thus (4.3) defines a vector field on N . The vector field (4.3) is usually denoted

$$\psi_* X = d\psi(X_{\psi^{-1}(p)})$$

and referred to as the *push-forward* of X .

We now wish to show that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where \mathfrak{g}_\pm are the ± 1 -eigenspaces of \mathcal{T} . This is proven by introducing an inner product $(X, Y)_\mathfrak{g}$ on \mathfrak{g} with respect to which \mathcal{T} is self-adjoint.

Theorem 4.2. *Consider \mathfrak{g} as a Hilbert space with inner product $(X, Y)_\mathfrak{g}$. The operator $\mathcal{T} : \mathfrak{g} \rightarrow \mathfrak{g}$ is self-adjoint with $\mathcal{T}^2 = I$; hence*

$$(4.4) \quad \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

as an orthogonal direct sum of Hilbert spaces, where \mathfrak{g}_\pm are the ± 1 -eigenspaces of \mathcal{T} . Further, $\partial_t \in \mathfrak{g}_-$ hence $\dim(\mathfrak{g}_-) \geq 1$. Elements of \mathfrak{g}_- have hermitian quantizations, while elements of \mathfrak{g}_+ have anti-hermitian quantizations.²

Proof. Write (4.2) as

$$(4.5) \quad \mathcal{T}(X) = \theta^{-1*} X \theta^* = \theta_* X.$$

Thus \mathcal{T} is the operator of push-forward by θ . The push-forward of a Killing field by an isometry is another Killing field, hence the range of \mathcal{T} is contained

²It is not the case that \mathfrak{g}_- consists only of ∂_t . In particular, $\dim(\mathfrak{g}_-) = 2$ for $M = \mathbb{H}_2$. It can occur that $\dim \mathfrak{g}_+ = 0$.

in \mathfrak{g} . Also, \mathcal{T} must have a trivial kernel since $\mathcal{T}^2 = I$, and this implies that \mathcal{T} is surjective. It follows from (4.5) that \mathcal{T} is a Hermitian operator on \mathfrak{g} . Hence \mathcal{T} is diagonalizable and has real eigenvalues which are square roots of 1. This establishes the decomposition (4.4). That elements of \mathfrak{g}_- have hermitian quantizations, while elements of \mathfrak{g}_+ have anti-hermitian quantizations follows from Theorem 3.3. \square

A *Cartan involution* is a Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{g}$ which squares to the identity. It follows from (4.2) that \mathcal{T} is a Lie algebra homomorphism; thus, Theorem 4.2 implies that \mathcal{T} is a Cartan involution of \mathfrak{g} . This implies that the eigenspaces $(\mathfrak{g}_+, \mathfrak{g}_-)$ form a *Cartan pair*, meaning that

$$(4.6) \quad [\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-, \quad \text{and} \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

Clearly \mathfrak{g}_+ is a subalgebra while \mathfrak{g}_- is not, and any subalgebra contained in \mathfrak{g}_- is abelian.

5. REFLECTION-INVARIANT AND REFLECTED ISOMETRIES

Let $G = \text{Iso}(M)$ denote the isometry group of M , as above. Then G has a \mathbb{Z}_2 subgroup containing $\{1, \theta\}$. This subgroup acts on G by conjugation, which is just the action $\psi \rightarrow \psi^\theta := \theta\psi\theta$. Conjugation is an (inner) automorphism of the group, so

$$(\psi\phi)^\theta = \psi^\theta\phi^\theta, \quad (\psi^\theta)^{-1} = (\psi^{-1})^\theta.$$

Definition 5.1. We say that $\psi \in G$ is **reflection-invariant** if

$$\psi^\theta = \psi,$$

and that ψ is **reflected** if

$$\psi^\theta = \psi^{-1}.$$

Let G_{RI} denote the subgroup of G consisting of reflection-invariant elements, and let G_R denote the subset of reflected elements.

Note that G_{RI} is the stabilizer of the \mathbb{Z}_2 action, hence a subgroup. An alternate proof of this proceeds using $G_{RI} = \exp(\mathfrak{g}_+)$. Although G_R is closed under the taking of inverses and does contain the identity, the product of two reflected isometries is no longer reflected unless they commute. Generally, the product of an element of G_R with an element of G_{RI} is neither an element of G_R nor of G_{RI} . The only isometry that is both reflection-invariant and reflected is θ itself. Thus we have:

$$G_R \cap G_{RI} = \{1, \theta\} \subset G_R \cup G_{RI} \subsetneq G.$$

Theorem 5.2. *Let G^0 denote the connected component of the identity in G . Then G^0 is generated by $G_R \cup G_{RI}$. (This is a form of the Cartan decomposition for G .)*

Proof. Since $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a direct sum of vector spaces (though not of Lie algebras), we have

$$G^0 = \langle \exp(\mathfrak{g}) \rangle = \langle \exp(\mathfrak{g}_+) \cup \exp(\mathfrak{g}_-) \rangle.$$

Choose bases $\{\xi_{\pm,i}\}_{i=1,\dots,n_{\pm}}$ for \mathfrak{g}_{\pm} respectively. Then we have:

$$G^0 = \langle \{\exp(s\xi_{+,i}) : 1 \leq i \leq n_+, s \in \mathbb{R}\} \cup \{\exp(s\xi_{-,j}) : 1 \leq j \leq n_-, s \in \mathbb{R}\} \rangle.$$

Furthermore, $\exp(s\xi_{-,i})$ is reflected, while $\exp(s\xi_{+,i})$ is reflection-invariant, completing the proof. \square

Corollary 5.3. *The Lie algebra of the subgroup G_{RI} is \mathfrak{g}_+ .*

To summarize, the isometry group of a static space-time can always be generated by a collection of n ($= \dim \mathfrak{g}$) one-parameter subgroups, each of which consists either of reflected isometries, or reflection-invariant isometries.

6. CONSTRUCTION OF UNITARY REPRESENTATIONS

6.1. Self-adjointness of Semigroups. In this section, we recall several known results on self-adjointness of semigroups. Roughly speaking, these results imply that if a one-parameter family S_α of unbounded symmetric operators satisfies a semigroup condition of the form $S_\alpha S_\beta = S_{\alpha+\beta}$, then under suitable conditions one may conclude essential self-adjointness.

A theorem of this type appeared in a 1970 paper of Nussbaum [14], who assumed that the semigroup operators have a common dense domain. The result was rediscovered independently by Fröhlich, who applied it to quantum field theory in several important papers [5, 3]. For our intended application to quantum field theory, it turns out to be very convenient to drop the assumption that $\exists a$ such that the S_α all have a common dense domain for $|\alpha| < a$, in favor of the weaker assumption that $\bigcup_{\alpha>0} D(S_\alpha)$ is dense.

A generalization of Nussbaum's theorem which allows the domains of the semigroup operators to vary with the parameter, and which only requires the *union* of the domains to be dense, was later formulated and two independent proofs were given: one by Fröhlich [4], and another by Klein and Landau [12]. The latter also used this theorem in their construction of representations of the Euclidean group and the corresponding analytic continuation to the Lorentz group [13].

In order to keep the present article self-contained, we first define symmetric local semigroups and then recall the refined self-adjointness theorem of Fröhlich, and Klein and Landau.

Definition 6.1. Let \mathcal{H} be a Hilbert space, let $T > 0$ and for each $\alpha \in [0, T]$, let S_α be a symmetric linear operator on the domain $\mathcal{D}_\alpha \subset \mathcal{H}$, such that:

- (i) $\mathcal{D}_\alpha \supset \mathcal{D}_\beta$ if $\alpha \leq \beta$ and $\mathcal{D} := \bigcup_{0 < \alpha \leq T} \mathcal{D}_\alpha$ is dense in \mathcal{H} ,
- (ii) $\alpha \rightarrow S_\alpha$ is weakly continuous,
- (iii) $S_0 = I$, $S_\beta(\mathcal{D}_\alpha) \subset \mathcal{D}_{\alpha-\beta}$ for $0 \leq \beta \leq \alpha \leq T$, and

(iv) $S_\alpha S_\beta = S_{\alpha+\beta}$ on $\mathcal{D}_{\alpha+\beta}$ for $\alpha, \beta, \alpha + \beta \in [0, T]$.

In this situation, we say that $(S_\alpha, \mathcal{D}_\alpha, T)$ is a *symmetric local semigroup*.

It is important that \mathcal{D}_α is *not* required to be dense in \mathcal{H} for each α ; the only density requirement is (i).

Theorem 6.2 ([12, 4]). *For each symmetric local semigroup $(S_\alpha, \mathcal{D}_\alpha, T)$, there exists a unique self-adjoint operator A such that³*

$$\mathcal{D}_\alpha \subset D(e^{-\alpha A}) \quad \text{and} \quad S_\alpha = e^{-\alpha A}|_{\mathcal{D}_\alpha} \quad \text{for all } \alpha \in [0, T].$$

Also, $A \geq -c$ if and only if $\|S_\alpha f\| \leq e^{c\alpha}\|f\|$ for all $f \in \mathcal{D}_\alpha$ and $0 < \alpha < T$.

6.2. Reflection-Invariant Isometries.

Lemma 6.3. *Let ψ be a reflection-invariant isometry and assume $\exists p \in \Omega_+$ such that $\psi(p) \in \Omega_+$. Then ψ preserves the positive-time subspace, i.e. $\psi(\Omega_+) \subseteq \Omega_+$.*

Proof. We first prove that $\psi(\Sigma) \subseteq \Sigma$. Suppose not; then $\exists p \in \Sigma$ with $\psi(p) \notin \Sigma$. Assume $\psi(p) \in \Omega_+$ (without loss of generality: we could repeat the same argument with $\psi(p) \in \Omega_-$). Then Ω_+ contains $(\theta\psi\theta)(p) = \theta\psi(p) \in \Omega_-$, a contradiction since $\Omega_- \cap \Omega_+ = \emptyset$. We used the fact that $\theta|_\Sigma = \text{id}$ so $\theta(p) = p$. Hence ψ restricts to an isometry of Σ . It follows that the restriction of ψ to $M' = M \setminus \Sigma$ is also an isometry. However, $M' = \Omega_- \sqcup \Omega_+$, where \sqcup denotes the disjoint union. Therefore $\psi(\Omega_+)$ is wholly contained in either Ω_+ or Ω_- , since ψ is a homeomorphism and so $\psi(\Omega_+)$ is connected. The possibility that $\psi(\Omega_+) \subseteq \Omega_-$ is ruled out by our assumption, so $\psi(\Omega_+) \subseteq \Omega_+$. \square

Lemma 6.3 has the immediate consequence that if $\xi \in \mathfrak{g}_+$ then the one-parameter group associated to ξ is positive-time-invariant. This result plays a key role in the proof of Theorem 6.4.

6.3. Construction of Unitary Representations. The rest of this section is devoted to proving that the theory of symmetric local semigroups can be applied to the quantized operators on \mathcal{H} corresponding to each of a set of 1-parameter subgroups of $G = \text{Iso}(M)$. The proof relies upon Lemma 6.3, and Theorems 3.3, 3.5 and 6.2.

Theorem 6.4. *Let (M, g_{ab}) be a quantizable static space-time. Let ξ be a Killing field which lies in \mathfrak{g}_+ or \mathfrak{g}_- , with associated one-parameter group of isometries $\{\phi_\alpha\}_{\alpha \in \mathbb{R}}$. Then there exists a densely-defined self-adjoint operator A_ξ on \mathcal{H} such that*

$$\widehat{\Gamma}(\phi_\alpha) = \begin{cases} e^{-\alpha A_\xi}, & \text{if } \xi \in \mathfrak{g}_- \\ e^{i\alpha A_\xi} & \text{if } \xi \in \mathfrak{g}_+. \end{cases}$$

³The authors of [4, 12] also showed that

$$\widehat{\mathcal{D}} := \bigcup_{0 < \alpha \leq S} \left[\bigcup_{0 < \beta < \alpha} S_\beta(\mathcal{D}_\alpha) \right], \quad \text{where } 0 < S \leq T,$$

is a *core* for A , i.e. $(A, \widehat{\mathcal{D}})$ is essentially self-adjoint.

Proof. First suppose that $\xi \in \mathfrak{g}_-$, which implies that the isometries ϕ_α are reflected, and so $\Gamma(\phi_\alpha)^+ = \Gamma(\phi_\alpha)$. Define

$$\Omega_{\xi,\alpha} := \phi_\alpha^{-1}(\Omega_+).$$

For all α in some neighborhood of zero, $\Omega_{\xi,\alpha}$ is a nonempty open subset of Ω_+ , and moreover, as $\alpha \rightarrow 0^+$, $\Omega_{\xi,\alpha}$ increases to fill Ω_+ with $\Omega_{\xi,0} = \Omega_+$. These statements follow immediately from the fact that, for each $p \in \Omega_+$, $\phi_\alpha(p)$ is continuous with respect to α , and ϕ_0 is the identity map.

Since $\phi_\alpha(\Omega_{\xi,\alpha}) \subseteq \Omega_+$, we infer that $\Gamma(\phi_\alpha)\mathcal{E}_{\Omega_{\xi,\alpha}} \subseteq \mathcal{E}_+$. By Theorem 3.5, $\Gamma(\phi_\alpha)$ has a quantization which is a symmetric operator on the domain

$$\mathcal{D}_{\xi,\alpha} := \Pi(\mathcal{E}_{\Omega_{\xi,\alpha}}).$$

Note that $\mathcal{D}_{\xi,\alpha}$ is not necessarily dense in \mathcal{H} .⁴ We now show that Theorem 6.2 can be applied.

Fix some positive constant a with $\Omega_{\xi,a}$ nonempty. Note that

$$\bigcup_{0 < \alpha \leq a} \Omega_{\xi,\alpha} = \Omega_+ \quad \Rightarrow \quad \bigcup_{0 < \alpha \leq a} \mathcal{E}_{\Omega_{\xi,\alpha}} = \mathcal{E}_+.$$

It follows that

$$\mathcal{D}_\xi := \bigcup_{0 < \alpha \leq a} \mathcal{D}_{\xi,\alpha}$$

is dense in \mathcal{H} . This establishes condition (i) of Definition 6.1, and the other conditions are routine verifications. Theorem 6.2 implies existence of a densely-defined self-adjoint operator A_ξ on \mathcal{H} , such that

$$\widehat{\Gamma}(\phi_\alpha) = \exp(-\alpha A_\xi) \quad \text{for all } \alpha \in [0, a].$$

This proves the theorem in case $\xi \in \mathfrak{g}_-$.

Now suppose that $\xi \in \mathfrak{g}_+$, implying that the isometries ϕ_α are reflection-invariant, and

$$\Gamma(\phi_\alpha)^+ = \Gamma(\phi_\alpha)^{-1} = \Gamma(\phi_{-\alpha}) \quad \text{on } \mathcal{E}.$$

Lemma 6.3 implies that $\Gamma(\phi_\alpha)\mathcal{E}_+ \subseteq \mathcal{E}_+$. By Theorem 3.3, $\Gamma(\phi_\alpha)$ has a quantization $\widehat{\Gamma}(\phi_\alpha)$ which is defined and satisfies

$$\widehat{\Gamma}(\phi_\alpha)^* = \widehat{\Gamma}(\phi_\alpha)^{-1}$$

on the domain $\Pi(\mathcal{E}_+)$, which is dense in \mathcal{H} by definition. In this case we do not need Theorem 6.2; for each α , $\widehat{\Gamma}(\phi_\alpha)$ extends by continuity to a one-parameter unitary group defined on all of \mathcal{H} (not only for a dense subspace). By Stone's theorem,

$$\widehat{\Gamma}(\phi_\alpha) = \exp(i\alpha A_\xi)$$

for A_ξ self-adjoint and for all $\alpha \in \mathbb{R}$. The proof is complete. \square

⁴Density of $\mathcal{D}_{\xi,\alpha}$ would be implied by a Reeh-Schlieder theorem, which we do not prove except in the free case. Theorem 6.2 removes the need for a Reeh-Schlieder theorem in this argument.

7. ANALYTIC CONTINUATION

Each Riemannian static space-time (M, g_{ab}) has a Lorentzian continuation M_{lor} , which we construct as follows. In adapted coordinates, the metric g_{ab} on M takes the form

$$(7.1) \quad ds^2 = \mathcal{F}(x)dt^2 + \mathcal{G}_{\mu\nu}(x)dx^\mu dx^\nu.$$

The analytic continuation $t \rightarrow -it$ of (7.1) is standard and gives a metric of Lorentz signature, $ds^2_{\text{lor}} = -\mathcal{F} dt^2 + \mathcal{G} dx^2$, by which we define the Lorentzian space-time M_{lor} . Einstein's equation $\text{Ric}_g = kg$ is preserved by the analytic continuation, but we do not use this fact anywhere in the present paper.

Let $\{\xi_i^{(\pm)} : 1 \leq i \leq n_\pm\}$ be bases of \mathfrak{g}_\pm , respectively. Let $A_i^{(\pm)} = A_{\xi_i^{(\pm)}}$ be the densely-defined self-adjoint operators on \mathcal{H} , constructed by Theorem 6.4. Let

$$(7.2) \quad U_i^{(\pm)}(\alpha) = \exp(i\alpha A_i^{(\pm)}), \quad \text{for } 1 \leq i \leq n_\pm$$

be the associated one-parameter unitary groups on \mathcal{H} .

We claim that the group generated by the $n = n_+ + n_-$ one-parameter unitary groups (7.2) is isomorphic to the identity component of

$$G_{\text{lor}} := \text{Iso}(M_{\text{lor}}),$$

the group of Lorentzian isometries. Since locally, the group structure is determined by its Lie algebra, it suffices to check that the generators satisfy the defining relations of $\mathfrak{g}_{\text{lor}} := \text{Lie}(G_{\text{lor}})$.

Since quantization of operators preserves multiplication, we have

$$(7.3) \quad X, Y, Z \in \mathfrak{g}, [X, Y] = Z \quad \Rightarrow \quad [\widehat{\Gamma}(X), \widehat{\Gamma}(Y)] = \widehat{\Gamma}(Z).$$

In what follows, we will use the notation $\widehat{\mathfrak{g}}_\pm$ for $\{\widehat{\Gamma}(X) : X \in \mathfrak{g}_\pm\}$.

Quantization converts the elements of \mathfrak{g}_- from skew operators into Hermitian operators; i.e. elements of $\widehat{\mathfrak{g}}_-$ are Hermitian on \mathcal{H} and hence, elements of $i\widehat{\mathfrak{g}}_-$ are skew-symmetric on \mathcal{H} . Thus $\widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_-$ is a Lie algebra represented by skew-symmetric operators on \mathcal{H} .

Theorem 7.1. *We have an isomorphism of Lie algebras:*

$$(7.4) \quad \mathfrak{g}_{\text{lor}} \cong \widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_-.$$

Proof. Let $M_{\mathbb{C}}$ be the manifold obtained by allowing the t coordinate to take values in \mathbb{C} . Define $\psi : M_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ by $t \mapsto -it$. Then $\mathfrak{g}_{\text{lor}}$ is generated by

$$\{\xi_i^{(+)}\}_{1 \leq i \leq n_+} \cup \{\eta_j\}_{1 \leq j \leq n_-}, \quad \text{where } \eta_j := i\psi^*(\xi_j^{(-)}).$$

It is possible to define a set of real structure constants f_{ijk} such that

$$(7.5) \quad [\xi_i^{(-)}, \xi_j^{(-)}] = \sum_{k=1}^{n_+} f_{ijk} \xi_k^{(+)}.$$

Applying ψ^* to both sides of (7.5), the commutation relations of $\mathfrak{g}_{\text{lor}}$ are seen to be

$$(7.6) \quad [\eta_i, \eta_j] = -f_{ijk}\xi_k^{(+)},$$

together with the same relations for \mathfrak{g}_+ as before. Now (7.3) implies that (7.6) are the precisely the commutation relations of $\widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_-$, completing the proof of (7.4). \square

Corollary 7.2. *Let (M, g_{ab}) be a quantizable static space-time. The unitary groups (7.2) determine a unitary representation of G^0_{lor} on \mathcal{H} .*

7.1. Conclusions. We have obtained the following conclusions. There is a unitary representation of the group G^0_{lor} on the physical Hilbert space \mathcal{H} of quantum field theory on the static space-time M . This representation maps the time-translation subgroup into the unitary group $\exp(itH)$, where the energy $H \geq 0$ is a positive, densely-defined self-adjoint operator corresponding to the Hamiltonian of the theory. The Hilbert space \mathcal{H} contains a ground state $\Psi_0 = \hat{1}$ which is such that $H\Psi_0 = 0$ and Ψ_0 is invariant under the action of all spacetime symmetries. We obtain these results via analytic continuation from the Euclidean path integral, under mild assumptions on the measure which should include all physically interesting examples. This is done without introducing the field operators; nonetheless, Theorems 3.3 and 3.5 do suffice to construct field operators. In the special case $M = \mathbb{R}^d$ with $G = \text{SO}(4)$, we obtain a unitary representation of the proper orthochronous Lorentz group, $G^0_{\text{lor}} = L^{\uparrow}_+ = \text{SO}^0(3, 1)$.

8. HYPERBOLIC SPACE AND ANTI-DE SITTER SPACE

Consider Euclidean quantum field theory on $M = \mathbb{H}^d$. The metric is

$$ds^2 = r^{-2} \sum_{i=1}^d dx_i^2,$$

where we define $r = x^d$ for convenience. The Laplacian is

$$(8.1) \quad \Delta_{\mathbb{H}^d} = (2-d)r \frac{\partial}{\partial r} + r^2 \Delta_{\mathbb{R}^d}.$$

The $d-1$ coordinate vector fields $\{\partial/\partial x^i : i \neq d\}$ are all static Killing fields, and any one of the coordinates x^i ($i \neq d$) is a satisfactory representation of time in this space-time. It is convenient to define $t = x^1$ as before, and to identify t with time.

The time-zero slice is $M_0 = \mathbb{H}^{d-1}$. From

$$\mathbb{H}^d = \{v \in \mathbb{R}^{d,1} \mid \langle v, v \rangle = -1, v_0 > 0\}$$

it follows that $\text{Isom}(\mathbb{H}^d) = O^+(d, 1)$ and the orientation-preserving isometry group is $SO^+(d, 1)$.

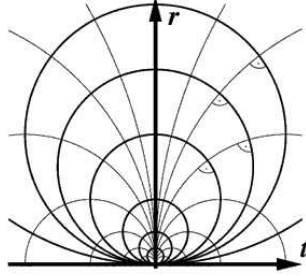


FIGURE 1. Flow lines of the Killing field $\zeta = (t^2 - r^2)\partial_t + 2tr \partial_r$ on \mathbb{H}^d .

For constant curvature spaces, one may solve Killing's equation $\mathcal{L}_K g = 0$ explicitly. Let us illustrate the solutions and their quantizations for $d = 2$. The three Killing fields

$$(8.2) \quad \xi = \partial_t, \quad \eta = t\partial_t + r\partial_r, \quad \zeta = (t^2 - r^2)\partial_t + 2tr \partial_r$$

are a convenient basis for \mathfrak{g} . Any d -dimensional manifold satisfies $\dim \mathfrak{g} \leq d(d+1)/2$, manifolds saturating the bound are said to be *maximally symmetric*, and \mathbb{H}^d is maximally symmetric.

Now, $\partial_t f(-t) = -f'(-t)$ so $\partial_t \Theta = -\Theta \partial_t$, hence $\partial_t \in \mathfrak{g}_-$. Similar calculations show $[\Theta, \eta] = 0$ and $\Theta \zeta = -\zeta \Theta$. Thus η spans \mathfrak{g}_+ , while ∂_t, ζ span \mathfrak{g}_- . The commutation relations⁵ for \mathfrak{g} are:

$$[\eta, \zeta] = \zeta, \quad [\eta, \partial_t] = -\partial_t, \quad [\zeta, \partial_t] = -2\eta.$$

These calculations verify that $(\mathfrak{g}_+, \mathfrak{g}_-)$ forms a Cartan pair, as defined in (4.6).

The flows associated to (8.2) are easily visualized: ξ is a right-translation, and η flow-lines are radially outward from the Euclidean origin. The flows of ζ are Euclidean circles, indicated by the darker lines in Figure 1. Hence the flows of η are defined on all of \mathcal{E}_+ , while the flows of ζ are analogous to space-time rotations in \mathbb{R}^2 , and hence, must be defined on a wedge of the form

$$W_\alpha = \{(t, r) : t, r > 0, \tan^{-1}(r/t) < \alpha\}.$$

The simple geometric idea of Section 6.2 is nicely confirmed in this case: the flows of η (the generator of \mathfrak{g}_+) preserve the $t = 0$ plane, and are separately isometries of Ω_+ and Ω_- .

Corollary 7.2 implies that the procedure outlined above defines a unitary representation of the identity component of $\text{Iso}(AdS_2)$ on the physical Hilbert space \mathcal{H} for quantum field theory on this background, including theories with interactions that preserve the symmetry. Since $\text{Iso}(AdS_{d+1}) =$

⁵Note that quite generally $[\mathfrak{g}_-, \mathfrak{g}_-] \subseteq \mathfrak{g}_+$ so it's automatic that $[\zeta, \partial_t]$ is proportional to η .

$\mathrm{SO}(d, 2)$, we have a unitary representation of $\mathrm{SO}^0(1, 2)$. The latter is a non-compact, semisimple real Lie group, and thus it has no finite-dimensional unitary representations, but a host of interesting infinite-dimensional ones.

APPENDIX A. EUCLIDEAN REEH-SCHLIEDER THEOREM

We prove the Euclidean Reeh-Schlieder property for free theories on curved backgrounds. It is reasonable to expect this property to extend to *interacting* theories on curved backgrounds, but it would have to be established for each such model since it depends explicitly on the two-point function.

The Reeh-Schlieder theorem guarantees the existence of a dense quantization domain based on any open subset of Ω_+ . For this reason, one could use the Reeh-Schlieder (RS) theorem with Nussbaum's theorem [14] to construct a second proof of Theorem 6.4 under the additional assumption that M is real-analytic.

Fortunately, our proof of Theorem 6.4 is completely independent of the Reeh-Schlieder property. This has two advantages: we do not have to assume M is a real-analytic manifold and, more importantly, our proof of Theorem 6.4 generalizes immediately and transparently to interacting theories as long as the Hilbert space \mathcal{H} is not modified by the interaction.

We state and prove this using the one-particle space; however, the result clearly extends to the quantum-field Hilbert space.

Theorem A.1. *Let M be a quantizable static space-time endowed with a real-analytic structure, and assume that g_{ab} is real-analytic. Let $\mathcal{O} \subset \Omega_+$ and $\mathcal{D} = C^\infty(\mathcal{O}) \subset L^2(\Omega_+)$. Then $\hat{\mathcal{D}}^\perp = \{0\}$.*

Proof. Let $f \in L^2(\Omega_+)$ with $\hat{f} \perp \mathcal{D}$. For $x \in \Omega_+$, define

$$\eta(x) := \langle \hat{f}, \hat{\delta}_x \rangle_{\mathcal{H}} = \langle \Theta f, C\delta_x \rangle_{L^2}.$$

Real-analyticity of $\eta(x)$ follows from the real-analyticity of (the integral kernel of) C , which in turn follows from the elliptic regularity theorem in the real-analytic category (see for instance [1, Sec. II.1.3]). Now by assumption, for any $g \in C_c^\infty(\mathcal{O})$, we have

$$0 = \langle \hat{g}, \hat{f} \rangle_{\mathcal{H}} = \langle \Theta f, Cg \rangle_{L^2(M)}.$$

Let $g \rightarrow \delta_x$ for $x \in \mathcal{O}$. Then $0 = \langle \Theta f, C\delta_x \rangle_{L^2} \equiv \eta(x)$. Since $\eta|_{\mathcal{O}} = 0$, by real-analyticity we infer the vanishing of η on Ω_+ , completing the proof. \square

Acknowledgements. We are grateful to Hanno Gottschalk and Alexander Strohmaier for helpful discussions, and G.R. is grateful to the Universität Bonn for their hospitality during February 2007.

REFERENCES

- [1] Lipman Bers, Fritz John, and Martin Schechter. *Partial differential equations*. American Mathematical Society, Providence, R.I., 1979. Lectures in Applied Mathematics 3.

- [2] N. D. Birrell and P. C. W. Davies. *Quantum fields in curved space*, volume 7 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, Cambridge, 1982.
- [3] W. Driessler and J. Fröhlich. The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 27(3):221–236, 1977.
- [4] J. Fröhlich. Unbounded, symmetric semigroups on a separable Hilbert space are essentially selfadjoint. *Adv. in Appl. Math.*, 1(3):237–256, 1980.
- [5] Jürg Fröhlich. The pure phases, the irreducible quantum fields, and dynamical symmetry breaking in Symanzik-Nelson positive quantum field theories. *Ann. Physics*, 97(1):1–54, 1976.
- [6] Stephen A. Fulling. *Aspects of quantum field theory in curved spacetime*, volume 17 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1989.
- [7] I. M. Gel'fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein.
- [8] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.
- [9] Arthur Jaffe. Constructive quantum field theory. In *Mathematical physics 2000*, pages 111–127. Imp. Coll. Press, London, 2000.
- [10] Arthur Jaffe. *Introduction to Quantum Field Theory*. 2005. Lecture notes from Harvard Physics 289r, available in part online at <http://www.arthurjaffe.com/Assets/pdf/IntroQFT.pdf>.
- [11] Arthur Jaffe and Gordon Ritter. Quantum field theory on curved backgrounds. i. the euclidean functional integral. *Comm. Math. Phys.*, 270(2):545–572, 2007.
- [12] Abel Klein and Lawrence J. Landau. Construction of a unique self-adjoint generator for a symmetric local semigroup. *J. Funct. Anal.*, 44(2):121–137, 1981.
- [13] Abel Klein and Lawrence J. Landau. From the Euclidean group to the Poincaré group via Osterwalder-Schrader positivity. *Comm. Math. Phys.*, 87(4):469–484, 1983.
- [14] A. E. Nussbaum. Spectral representation of certain one-parametric families of symmetric operators in Hilbert space. *Trans. Amer. Math. Soc.*, 152:419–429, 1970.
- [15] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. *Comm. Math. Phys.*, 31:83–112, 1973.
- [16] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. II. *Comm. Math. Phys.*, 42:281–305, 1975. With an appendix by Stephen Summers.

- [17] Barry Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.
- [18] Robert M. Wald. Quantum field theory in curved space-time. In *Gravitation et quantifications (Les Houches, 1992)*, pages 63–167. North-Holland, Amsterdam, 1995.

E-mail address: `arthur_jaffe@harvard.edu`

HARVARD UNIVERSITY, 17 OXFORD ST., CAMBRIDGE, MA 02138

E-mail address: `ritter@post.harvard.edu`

HARVARD UNIVERSITY, 17 OXFORD ST., CAMBRIDGE, MA 02138