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Growth and Hölder conditions for the sample paths of Feller processes

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Abstract. Let $(A, D(A))$ be the infinitesimal generator of a Feller semigroup such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n)$ is a pseudo-differential operator with symbol $-p(x,\xi)$ satisfying $\|p(\bullet,\xi)\|_\infty \le c(1+\|\xi\|^2)$ and $|\operatorname{Im} p(x,\xi)| \le c_0 \operatorname{Re} p(x,\xi)$. We show that the associated Feller process $\{X_t\}_{t\ge 0}$ on $\mathbb{R}^n$ is a semimartingale, even a homogeneous diffusion with jumps (in the sense of [21]), and characterize the limiting behaviour of its trajectories as $t \to 0$ and ∞. To this end, we introduce various indices, e.g., $\beta_\infty^x := \inf\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \sup_{\|x-y\|\le 2/\|\xi\|} |p(y,\xi)|/\|\xi\|^\lambda = 0\}$ or $\delta_\infty^x := \inf\{\lambda > 0 : \liminf_{\|\xi\|\to\infty} \inf_{\|x-y\|\le 2/\|\xi\|} \sup_{\|\epsilon\|\le 1} |p(y, \|\xi\|\epsilon)|/\|\xi\|^\lambda = 0\}$, and obtain a.s. $(\mathbb{P}^x)$ that $\lim_{t\to 0} t^{-1/\lambda} \sup_{s\le t} \|X_s - x\| = 0$ or ∞ according to $\lambda > \beta_\infty^x$ or $\lambda < \delta_\infty^x$. Similar statements hold for the limit inferior and superior, and also for $t \to \infty$. Our results extend the constant-coefficient (i.e., Lévy) case considered by W. Pruitt [27].

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1 Introduction

The aim of this paper is to characterize the global growth properties and local Hölder behaviour for the trajectories of a large class of Feller processes. Our investigations were very much influenced by the paper [27] by W. Pruitt on *The growth of random walks and Lévy processes*, which are particularly simple examples of Feller processes. In terms of their infinitesimal generators, Lévy processes belong to operators with *constant coefficients* whereas we are interested in general Feller processes admitting generators with *variable coefficients*. The results, however, are quite similar: the limiting behaviour of the process as $t \to 0$ or $t \to \infty$ is governed by the limiting behaviour of the *symbol* of the generator in the co-variable as $\|\xi\| \to \infty$ or $\|\xi\| \to 0$. The latter is usually expressed by certain indices, first introduced by Blumenthal & Getoor [4] for Lévy processes, see [28] for an up-to-date survey. Admittedly, our approach is technical and lacks the elegance of the arguments in [27], but the fact that we do allow variable coefficients seems to require semimartingale techniques on the one hand, and methods from the theory of pseudo-differential operators on the other.

A *Feller process* $\{X_t\}_{t\geq 0}$ with state space $\mathbb{R}^n$ is a strong Markov process such that the associated operator semigroup $\{T_t\}_{t\geq 0}$, defined on the continuous functions vanishing at infinity via

$$T_t u(x) = \mathbb{E}^x(u(X_t)), \quad u \in C_\infty(\mathbb{R}^n), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

enjoys the *Feller property*, i.e., is a *Feller semigroup*. We can always choose a càdlàg version of the process, and we will do so without further notice. Note that (1.1) establishes a correspondence between Feller processes and Feller semigroups.

A *Feller semigroup* is a one-parameter family of operators $T_t : C_\infty(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n)$, such that $T_t \circ T_s = T_{t+s}$, $\lim_{t\to 0} \|T_t u - u\|_\infty = 0$, and $0 \leq T_t u \leq 1$ whenever $0 \leq u \leq 1$. As usual, the (*infinitesimal*) *generator* $(A, D(A))$ is given by the strong limit

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t}$$

on the set $D(A) \subset C_\infty(\mathbb{R}^n)$ of those $u \in C_\infty(\mathbb{R}^n)$ where this limit exists in norm sense. We will call $(A, D(A))$ *Feller generator*, for short.

Write $\hat{u}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx$ for the Fourier transform. Under the assumption that the test functions $C_c^\infty(\mathbb{R}^n)$ are contained in $D(A)$, Ph. Courrège [6] – see also the closely related papers [36] by W. von Waldenfels and (in a somewhat different context) [5] by J.-M. Bony, Ph. Courrège, and P. Priouret – proved that the generator A (restricted to $C_c^\infty(\mathbb{R}^n)$) is a *pseudo-differential operator*,

$$Au(x) = -p(x, D)u(x)$$
$$:= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)\, d\xi, \quad u \in C_c^\infty(\mathbb{R}^n) \ , \tag{1.2}$$

with *symbol* $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. The symbol is locally bounded in (x, ξ), measurable as a function of x, and for every fixed $x \in \mathbb{R}^n$ it is a *continuous negative definite function* in the co-variable. This is to say that it enjoys the following *Lévy-Khinchine representation*,

$$p(x, \xi) = a(x) - i\ell(x) \cdot \xi + \xi \cdot Q(x)\xi$$
$$+ \int_{y\neq 0} \left(1 - e^{-iy\cdot\xi} - \frac{iy \cdot \xi}{1 + \|y\|^2}\right) N(x, dy) \tag{1.3}$$

where $(a(x), \ell(x), Q(x), N(x, dy))$ are the usual *Lévy characteristics*, i.e. $a(x) \geq 0$, $\ell(x) \in \mathbb{R}^n$, $Q(x) \in \mathbb{R}^{n\times n}$ positive semi-definite, and a (measurable) kernel $N(x, \cdot)$ on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{y\neq 0} \frac{\|y\|^2}{1+\|y\|^2} N(x, dy) < \infty$. Equivalently, "$p(x, 0) \geq 0$" and "$\xi \mapsto e^{-tp(x,\xi)}$ is positive definite" (in the usual sense) could have served as definiton. This shows that Lévy processes – they are given by convolution semigroups – are exactly those processes which are generated by constant-coefficient pseudo-differential operators. Their symbols are obviously given by the characteristic exponents (i.e., logarithms of the characteristic functions) of the Lévy processes.

This relation is no longer true for general Feller processes. But, if the domain of the generator is sufficiently rich (e.g., contains the test functions) we still have

$$\frac{d}{dt} \mathbb{E}^x (e^{-i(X_t - x)\cdot\xi}) \Big|_{t=0} = -p(x, \xi), \quad x, \xi \in \mathbb{R}^n \ ,$$

i.e., the symbol can be interpreted probabilistically as derivative of the characteristic function of the process. Under some mild additional assumptions, this is proved in [17], for the general case see [31, 33].

Many properties of negative definite functions and convolution semigroups are discussed in the monograph [3] by C. Berg and G. Forst. We will mention only two further facts, the – at most – quadratic growth of negative definite functions, which reads in our setting

$$|p(x_0, \xi)| \leq 2 \sup_{\|\epsilon\|\leq 1} p(x_0, \epsilon)(1 + \|\xi\|^2), \quad \xi \in \mathbb{R}^n, \quad x_0 \in \mathbb{R}^n \ , \tag{1.4}$$

and the generalized *Peetre inequality*

$$(1 + |p(x_0, \xi \pm \eta)|) \leq 2(1 + |p(x_0, \xi)|)(1 + |p(x_0, \eta)|),$$
$$\xi, \eta \in \mathbb{R}^n, x_0 \in \mathbb{R}^n \ . \tag{1.5}$$

Both are simple consequences of the subadditivity of $\xi \mapsto \sqrt{|p(x_0, \xi)|}$ as is true for any negative definite function. In this form, Peetre's inequality is due to W. Hoh [8].

Two other assumptions on the symbol turn out to be important for our study,

$$\begin{aligned} &\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2), \quad \text{and sometimes also} \\ &|\operatorname{Im} p(x,\xi)| \leq c_0 \operatorname{Re} p(x,\xi) \ . \end{aligned} \tag{1.6}$$

The first of them should be read as *there are only bounded coefficients* (compare in this context Lemma 2.1), whereas the second reminds of some kind of *sector condition* (cf. [2] in the constant-coefficient case). We conjecture that there is a strong link between the latter estimate and the analyticity of the semigroup generated by such an operator $-p(x,D)$. Probabilistically, this estimate means that there is no dominating drift term.

Recently, several authors investigated the converse problem: which (additional) assumptions on the symbol $-p(x,\xi)$ are sufficient in order that $-p(x,D)$ (without dominating quadratic part, say) extends to a Feller generator? Purely analytic constructions are given in [23, 13, 14, 25, 15, 10, 22], while the papers [1, 35, 8, 9] rely on the martingale problem approach sketched out by Stroock [34]. A good survey on these topics is given in [16]. Since we do not need these constructions explicitly, we will not state the conditions imposed on $p(x,\xi)$ in greater detail. However, we should like to mention that almost all of these papers do assume our conditions (1.6), which are thus not too restrictive to produce no interesting or only trivial examples.

Let us give a brief outline of how our paper is organized. Section 2 deals with the *size* of the domain of the (extended) Feller generator. Essentially, Theorem 2.6 states that $C^2_\infty(\mathbb{R}^n) \subset D(A)$ and that A extends reasonably to $C^2_b(\mathbb{R}^n)$. Theorem 2.7 exhibits another large subset of $D(A)$ which is interesting on its own. It can, however, also be used to show that our results on trajectories of Feller processes obtained in [31, 32, 30] do hold in greater generality than originally stated there. In Section 3 the results of the previous section will be used to show that the Feller process $\{X_t\}_{t\geq 0}$ generated by $(A, D(A))$ is a homogeneous diffusion with jumps (in the sense of Jacod & Shiraev [21]) and that its semimartingale characteristics coincide – essentially – with the Lévy characteristics of the symbol.

The main results of the paper are contained in Theorems 4.3 and 4.6 of Section 4. We use classical Borel-Cantelli techniques in order to characterize the limiting behaviour of expressions of the type

$$\liminf_{t\to 0}/\limsup_{t\to 0} t^{-1/\lambda} \sup_{s\le t} \|X_s - x\| \quad \text{and}$$
$$\liminf_{t\to\infty}/\limsup_{t\to\infty} t^{-1/\lambda} \sup_{s\le t} \|X_s - x\| \ .$$

(Sharp) bounds for the exponent λ are given in terms of various indices "β", "δ" (see Definitions 4.2, 4.5) which are determined by the asymptotics of the symbol in the co-variable. An application to first passage times, Theorem 4.7, is included. Much nicer descriptions of the indices are obtained in Section 5 where it is also shown, that our β's and δ's are really generalizations of the Blumenthal-Getoor and Pruitt indices. For stable-like processes with symbols $\|\xi\|^{a(x)}$ in the sense of Bass [1], all indices describing the local Hölder properties coincide and equal $a(x)$.

The last section is rather technical. There we prove the key Lemma 4.1 used in the fourth section:

$$\mathbb{P}^x\Big(\sup_{s\le t} \|X_s - x\| \ge R\Big) \le c_n t H(x,R) \quad \text{and}$$
$$\mathbb{P}^x\Big(\sup_{s\le t} \|X_s - x\| < R\Big) \le \frac{c}{th(x,R)}$$

where H and h are functions that are given in terms of the symbol $p(x,\xi)$. Our proof requires semimartingale techniques which are accessible through the results of Section 3. Strictly speaking, Lemma 4.1 should be viewed as the main *technical* result of this paper.

Notation. $C_c(\mathbb{R}^n), C_\infty(\mathbb{R}^n), C_b(\mathbb{R}^n)$ denote the continuous functions with compact support, which vanish at infinity, or which are bounded, and $B_b(\mathbb{R}^n)$ are the bounded Borel measurable functions; superscripts refer to differentiability properties. $\hat{u}(\xi) = \int e^{-ix\xi} u(x) \,đx$ is the Fourier transform, $đx := (2\pi)^{-n/2} dx$ means normalized Lebesgue measure, $\delta_y(dx)$ the Dirac measure (unit mass) at y, and $\mathbb{E}^x(\bullet) = \int \bullet \, d\mathbb{P}^x$ stands for the expectation; the superscript x indicates that the process starts a.s. at x. When dealing with random variables we often suppress ω. We write $a \wedge b$ and $a \vee b$ for the minimum and maximum of a, b, and $a \sim b$ means that the ratio of the two sides is bounded above and below by finite, strictly positive constants. All other notations are standard or should be clear from the context.

2 The domain of a Feller generator and the extended generator

Let $(A, D(A))$ be the generator of a Feller semigroup. As we have seen, the condition $C_c^\infty(\mathbb{R}^n) \subset D(A)$ ensures that the restricion $A|C_c^\infty(\mathbb{R}^n)$ is a pseudo differential operator $-p(x,D)$ with symbol $-p(x,\xi)$. The function $p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ is locally bounded (in both variables), measurable as a function of x, and, for fixed $x\in\mathbb{R}^n$, continuous negative definite, i.e., has a Lévy-Khinchine representation (1.3).

As in the constant-coefficient case, cf. [3, Corollary 7.16], the subadditivity of the function $\xi\mapsto\sqrt{|p(x,\xi)|}$ implies

$$|p(x,\xi)|\le 2\sup_{\|\eta\|\le 1}|p(x,\eta)|(1+\|\xi\|^2)\ . \tag{2.1}$$

We will frequently assume that $\sup_{x\in\mathbb{R}^n}\sup_{\|\eta\|\le 1}|p(x,\eta)|<\infty$, i.e., that (2.1) holds uniformly in x. The following Lemma gives a condition for this in terms of the Lévy characteristics of $p(x,\xi)$.

Lemma 2.1 *Let* $p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ *be given by the Lévy-Khinchine formula* (1.3) *with Lévy characteristics* $(a(x),\ell(x),Q(x),N(x,dy))$. *Then*

$$\sup_{x\in\mathbb{R}^n}|p(x,\xi)|\le c(1+\|\xi\|^2),\quad \xi\in\mathbb{R}^n\ , \tag{2.2}$$

if and only if

$$\|a\|_\infty+\|\ell\|_\infty+\|Q\|_\infty+\left\|\int_{y\ne 0}\frac{\|y\|^2}{1+\|y\|^2}N(\bullet,dy)\right\|_\infty<\infty\ . \tag{2.3}$$

Proof. By Taylor's formula we get for $\|y\|\le 1$

$$\begin{aligned}\left|1-\mathrm{e}^{-\mathrm{i}y\cdot\xi}-\frac{\mathrm{i}y\cdot\xi}{1+\|y\|^2}\right|&\le\left|1-\mathrm{e}^{-\mathrm{i}y\cdot\xi}-\mathrm{i}y\cdot\xi\right|+\left|y\cdot\xi-\frac{y\cdot\xi}{1+\|y\|^2}\right|\\&\le\frac{1}{2}\|y\|^2\|\xi\|^2+\frac{\|y\|^3\|\xi\|}{1+\|y\|^2}\le\frac{\|y\|^2}{1+\|y\|^2}(\|\xi\|^2+\|\xi\|)\ ,\end{aligned}$$

and for $\|y\|>1$,

$$\left|1-\mathrm{e}^{-\mathrm{i}y\cdot\xi}-\frac{\mathrm{i}y\cdot\xi}{1+\|y\|^2}\right|\le 2+\frac{\|y\|\|\xi\|}{1+\|y\|^2}\le\frac{\|y\|^2}{1+\|y\|^2}(4+\|\xi\|)\ .$$

This gives

$$\left|1-\mathrm{e}^{-\mathrm{i}y\cdot\xi}-\frac{\mathrm{i}y\cdot\xi}{1+\|y\|^2}\right|\le 4\frac{\|y\|^2}{1+\|y\|^2}(1+\|\xi\|^2),\quad y,\xi\in\mathbb{R}^n\ . \tag{2.4}$$

It is easy to see, cf. [20, Lemma 5.2, (5.4)], that

$$\frac{\|y\|^2}{1+\|y\|^2}=\int_{\eta\ne 0}(1-\cos y\cdot\eta)g_n(\eta)\,d\eta \tag{2.5}$$

is a continuous negative definite function whose Lévy measure has the density

$$g_n(\eta) = \frac{1}{2}\int_0^\infty (2\pi\rho)^{-n/2} e^{-\|\eta\|^2/(2\rho)} e^{-\rho/2}\, d\rho, \qquad \eta \in \mathbb{R}^n \ . \tag{2.6}$$

Obviously, $g_n(\eta)$ possesses all moments.

Assume that (2.5) is satisfied. Using (2.5) we see

$$\begin{aligned}
\int_{y\neq 0} \frac{\|y\|^2}{1+\|y\|^2} N(x,dy) &= \int_{y\neq 0}\int_{\eta\neq 0} (1-\cos y\cdot\eta) g_n(\eta)\, d\eta\, N(x,dy) \\
&= \int_{\eta\neq 0} \big(\operatorname{Re} p(x,\eta) - a(x) - \eta\cdot Q(x)\eta\big) g_n(\eta)\, d\eta \\
&\leq \int_{\eta\neq 0} \operatorname{Re} p(x,\eta) g_n(\eta)\, d\eta \\
&\leq c\int_{\eta\neq 0} (1+\|\eta\|^2) g_n(\eta)\, d\eta < \infty
\end{aligned}$$

uniformly for all $x \in \mathbb{R}^n$. Moreover, $a(x) = p(x,0) \leq c$ and from (2.4) we get

$$\begin{aligned}
|\ell(x)\cdot\xi| &\leq |\operatorname{Im} p(x,\xi)| + \left|\operatorname{Im}\int_{y\neq 0}\left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+\|y\|^2}\right) N(x,dy)\right| \\
&\leq \left(c + 4\int_{y\neq 0} \frac{\|y\|^2}{1+\|y\|^2} N(x,dy)\right)(1+\|\xi\|^2)
\end{aligned}$$

uniformly in $x \in \mathbb{R}^n$, thus $\|\ell\|_\infty < \infty$. Finally,

$$\begin{aligned}
|\xi\cdot Q(x)\xi| \leq{}& |p(x,\xi)| + a(x) + |\ell(x)\cdot\xi| \\
&+ 4\int_{y\neq 0} \frac{\|y\|^2}{1+\|y\|^2} N(x,dy)(1+\|\xi\|^2)
\end{aligned}$$

and $\|Q\|_\infty < \infty$ follows with the preceding calculations.

Conversely, let (2.3) be valid. From $\|a\|_\infty + \|\ell\|_\infty + \|Q\|_\infty < \infty$ we get

$$\sup_{x\in\mathbb{R}^n} \big|a(x) - i\ell(x)\cdot\xi + \xi\cdot Q(x)\xi\big| \leq c(1+\|\xi\|^2)$$

and (2.4) gives

$$\begin{aligned}
&\left|\int_{y\neq 0}\left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+\|y\|^2}\right) N(x,dy)\right| \\
&\qquad \leq 4\int_{y\neq 0} \frac{\|y\|^2}{1+\|y\|^2} N(x,dy)(1+\|\xi\|^2)
\end{aligned}$$

uniformly in $x \in \mathbb{R}^n$.

Remark 2.2 The statement of the above Lemma has an obvious extension for uniform convergence on compact sets. Note that in this case the condition $\|p(\bullet,\xi)\|_\infty \le c(1+\|\xi\|^2)$ can be replaced by "$(x,\xi)\mapsto p(x,\xi)$ *is locally bounded*" because

$$\sup_{\|x\|\le R} |p(x,\xi)| \le 2 \sup_{\|x\|\le R}\sup_{\|\eta\|\le 1} |p(x,\eta)|(1+\|\xi\|^2) \le c_R(1+\|\xi\|^2)\ .$$

In order to study the domain $D(A)$ of the Feller generator A, it is often useful to rewrite $-p(x,D)$ as *integro-differential operator*. Using the Lévy-Khinchine representation (1.3) for the symbol we get for $u\in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}-p(x,D)u(x) = \int_{\mathbb{R}^n}\Bigg[&\mathrm{i}\ell(x)\cdot\xi - a(x) - \xi\cdot Q(x)\xi\\ &- \int_{y\neq 0}\left(1-\mathrm{e}^{-\mathrm{i}y\cdot\xi} - \frac{\mathrm{i}y\cdot\xi}{1+\|y^2\|}\right)N(x,dy)\Bigg]\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,đ\xi\\ = \ell(x)\cdot\nabla u(x) &- a(x)u(x) + \sum_{j,k=1}^n q_{jk}(x)\partial_j\partial_k u(x)\\ &+ \int_{y\neq 0}\int_{\mathbb{R}^n}\left(\mathrm{e}^{-\mathrm{i}y\cdot\xi} - 1 + \frac{\mathrm{i}y\cdot\xi}{1+\|y\|^2}\right)\hat{u}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,đ\xi\, N(x,dy)\ ,\end{aligned}$$

where the change of the order of integration is justified since $u\in C_c^\infty(\mathbb{R}^n)$ and by the estimate (2.4). Thus, $-p(x,D) = I(p)|C_c^\infty(\mathbb{R}^n)$ where $I(p)$ is given by

$$\begin{aligned}I(p)u(x) :=& -a(x)u(x) + \ell(x)\cdot\nabla u(x) + \sum_{j,k=1}^n q_{jk}(x)\partial_j\partial_k u(x)\\ &+ \int_{y\neq 0}\left(u(x-y) - u(x) + \frac{y\cdot\nabla u(x)}{1+\|y^2\|}\right)N(x,dy),\\ &\qquad u\in C_b^2(\mathbb{R}^n)\ . \end{aligned}\tag{2.7}$$

The next Lemma shows that $-p(x,D)$ determines the Feller generator $(A,D(A))$ on a large subset of $C_\infty(\mathbb{R}^n)$. In general, $C_c^\infty(\mathbb{R}^n)$ is not a core for A, i.e., there can be many extensions of $-p(x,D)$ to a Feller generator. One of those is, of course, $(A,D(A))$.

Lemma 2.3 *Let $(A,D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n)\subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$. Assume that the symbol satisfies $\|p(\bullet,\xi)\|_\infty \le c(1+\|\xi\|^2)$. Then $-p(x,D)$ can be extended to a Feller* *generator and for any such extension, say,* $\big(-\widetilde{p(x,D)}, D(\widetilde{p(x,D)})\big)$ *one has*

$$C^2_\infty(\mathbb{R}^n) \subset D(\widetilde{p(x,D)}) \ . \tag{2.8}$$

Moreover,

$$\|p(\bullet,D)u\|_\infty \le c \sum_{|\alpha|\le 2} \|\partial^\alpha u\|_\infty \tag{2.9}$$

holds for $u \in C^\infty_c(\mathbb{R}^n)$ *and extends for* $I(p)$ *to* $C^2_b(\mathbb{R}^n)$.

This Lemma seems to be some kind of *folklore*, at least if one claims $C^\infty_c(\mathbb{R}^n) \subset D(A)$, see [29, VII (1.13) & notes to Chapter VII]; however, no proof is given there.

Proof. Since $-p(x,D) \subset A$, $-p(x,D)$ does have extensions to a Feller generator. Now (2.8) follows from (2.9) since any such extension, say $-\widetilde{p(x,D)}$, is a closed operator: for *every* sequence $\{u_k\}_{k\in\mathbb{N}}$, $u_k \in C^\infty_c(\mathbb{R}^n) \subset D(\widetilde{p(x,D)})$ such that $u_k \to u$ in $C^2_\infty(\mathbb{R}^n)$, the functions $-p(x,D)u_k(x) = -\widetilde{p(x,D)}u_k(x)$ are by (2.9) a Cauchy sequence (w.r.t. uniform convergence), hence $u \in D(\widetilde{p(x,D)})$ by the closedness of $-\widetilde{p(x,D)}$ and

$$C^2_\infty(\mathbb{R}^n) = \overline{C^\infty_c(\mathbb{R}^n)}^{\sum_{|\alpha|\le 2}\|\partial^\alpha \bullet\|_\infty} \subset D(\widetilde{p(x,D)}) \ .$$

For the proof of (2.9) we may assume that $a \equiv 0$, $\ell \equiv 0$, and $Q \equiv 0$ in the Lévy characteristics of $p(x,\xi)$. By (2.7) and a Taylor expansion we have for $u \in C^\infty_c(\mathbb{R}^n)$

$$\begin{aligned}
|p(x,D)u(x)| &= \left| \int_{y\neq 0} \left(u(x-y) - u(x) + \frac{y\cdot\nabla u(x)}{1+\|y\|^2}\right) N(x,dy) \right| \\
&\le \int_{0<\|y\|\le 1} \Bigg(\big| u(x-y) - u(x) + y\cdot\nabla u(x) \big| \\
&\qquad + \left| \frac{y\cdot\nabla u(x)}{1+\|y\|^2} - y\cdot\nabla u(x) \right| \Bigg) N(x,dy) \\
&\qquad + \int_{\|y\|>1} \left| u(x-y) - u(x) + \frac{y\cdot\nabla u(x)}{1+\|y\|^2} \right| N(x,dy) \\
&\le \int_{0<\|y\|\le 1} \Bigg(\sum_{j,k=1}^n \|\partial_j\partial_k u\|_\infty |y_j y_k| \\
&\qquad + \frac{\|y\|^3}{1+\|y\|^2} \|\nabla u\|_\infty \Bigg) N(x,dy)
\end{aligned}$$

$$+\int_{\|y\|>1}\left(2\|u\|_\infty+\frac{\|y\|}{1+\|y^2\|}\|\nabla u\|_\infty\right)N(x,dy)$$
$$\le c_n\sum_{|\alpha|\le 2}\|\partial^\alpha u\|_\infty\int_{y\neq 0}\frac{\|y\|^2}{1+\|y\|^2}N(x,dy)\ .$$

By Lemma 2.1 the latter expression is uniformly bounded in $x\in\mathbb{R}^n$, and the assertion follows. □

In fact, the above proof shows $\overline{p(x,D)}^{\|\bullet\|_\infty}|C^2_\infty(\mathbb{R}^n)=\widetilde{p(x,D)}|C^2_\infty(\mathbb{R}^n)$. Since on $C_c^\infty(\mathbb{R}^n)$ the operators $-p(x,D)$ and $I(p)$ (given by (2.7)) coincide, we obtain the following corollary.

Corollary 2.4 *In the setting of Lemma* 2.3 *any extension* $-\widetilde{p(x,D)}$ *of* $-p(x,D)$ *to a Feller generator satisfies* $-\widetilde{p(\bullet,D)}u=I(p)u$ *for* $u\in C^2_\infty(\mathbb{R}^n)$. *Here,* $I(p)$ *is the integro-differential operator given by* (2.7)

Lemma 2.5 *Let* $(A,D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^n)\subset D(A)$ *and* $A|C_c^\infty(\mathbb{R}^n)=-p(x,D)$. *Assume that the symbol satisfies* $\|p(\bullet,\xi)\|_\infty\le c(1+\|\xi\|^2)$. *Denote by* $-\widetilde{p(x,D)}$ *any extension of* $-p(x,D)$ *to a Feller generator. Then every sequence* $\{u_k\}_{k\in\mathbb{N}}$ *such that*

$$\begin{cases} u_k\in C_c^2(\mathbb{R}^n), & \sup_{k\in\mathbb{N}}\sum_{|\alpha|\le 2}\|\partial^\alpha u_k\|_\infty<\infty \\ \lim_{k\to\infty}u_k=u & \textit{locally uniformly in } C_b^2(\mathbb{R}^n)\end{cases}\tag{2.10}$$

satisfies $\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{N}}|\widetilde{p(x,D)}u_k(x)|<\infty$ *and* $\lim_{k\to\infty}\widetilde{p(x,D)}u_k(x)$ *exists.*

Proof. Corollary 2.4 shows that $-\widetilde{p(x,D)}u=I(p)u$ on $C_c^2(\mathbb{R}^n)$. From (2.10) we deduce the boundedness and convergence of the local part,

$$-a(\bullet)u_k+\ell(\bullet)\cdot\nabla u_k+\sum_{j,k=1}^n q_{ij}(\bullet)\partial_i\partial_j u_k$$
$$\to -a(\bullet)u+\ell(\bullet)\cdot\nabla u+\sum_{j,k=1}^n q_{ij}(\bullet)\partial_i\partial_j u$$

as $k\to\infty$. In order to see the convergence of the integral term we use dominated convergence. Note that by Taylor's theorem

$$\left|u_k(x-y)-u_k(x)+\frac{y\cdot\nabla u_k(x)}{1+\|y^2\|}\right|\le c\frac{\|y\|^2}{1+\|y\|^2}\sum_{|\alpha|\le 2}\|\partial^\alpha u_k\|_\infty$$

and that the right-hand side is uniformly bounded for all $k\in\mathbb{N}$. For fixed $x\in\mathbb{R}^n$ we find

$$\int_{y\neq 0}\left(u_k(x-y)-u_k(x)+\frac{y\cdot\nabla u_k(x)}{1+\|y\|^2}\right)N(x,dy)$$

$$\to\int_{y\neq 0}\left(u(x-y)-u(x)+\frac{y\cdot\nabla u(x)}{1+\|y\|^2}\right)N(x,dy)$$

as $k\to\infty$.

The asserted equi-boundedness of $p(\widetilde{x,D})u_k(x)$ in k and x follows with exactly the same calculations that were used to obtain (2.9). $\square$

Let $(A,D(A))$ be the generator of a Feller semigroup $\{T_t\}_{t\geq 0}$. Following Ethier and Kurtz [7, p. 23] we call the subset $\mathrm{EX}(A)\subset B_b(\mathbb{R}^n)\times B_b(\mathbb{R}^n)$,

$$\mathrm{EX}(A):=\left\{(f,g)\in B_b(\mathbb{R}^n)\times B_b(\mathbb{R}^n):T_tf-f=\int_0^t T_sg\,ds\right\}\quad(2.11)$$

the *extended generator* of $\{T_t\}_{t\geq 0}$. Clearly, the graph of A is contained in $\mathrm{EX}(A)$,

$$(f,Af)\in\mathrm{EX}(A)\quad\text{for all}\quad f\in D(A)\ .$$

Note that $\mathrm{EX}(A)$ is closed under *bounded pointwise limits* (*bp*-lim, for short), which are defined by

$$u=bp\text{-}\lim_{k\to\infty}u_k\quad\text{if}\quad\sup_{k\in\mathbb{N}}\|u_k\|_\infty<\infty\text{ and }u(x)=\lim_{k\to\infty}u_k(x),\quad x\in\mathbb{R}^n\ .$$

Theorem 2.6 *Let $(A,D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n)\subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n)=-p(x,D)$. Assume that the symbol satisfies $\|p(\bullet,\xi)\|_\infty\leq c(1+\|\xi\|^2)$. For every extension $-p(\widetilde{x,D})$ of $-p(x,D)$ to a Feller generator $C_\infty^2(\mathbb{R}^n)\subset D(p(\widetilde{x,D}))$ is satisfied. If $I(p)$ is given by* (2.7), we have $-p(\widetilde{x,D})|C_\infty^2(\mathbb{R}^n)=I(p)|C_\infty^2(\mathbb{R}^n)$ *and*

$$(u,I(p)u)\in\ \mathrm{EX}\big(-p(\widetilde{x,D})\big),\quad u\in C_b^2(\mathbb{R}^n)\ ,\qquad(2.12)$$

where $\mathrm{EX}\big(-p(\widetilde{x,D})\big)$ *denotes the extended generator.*

Proof. The first two claims follow from Lemma 2.3 and Corollary 2.4, respectively. In order to see (2.12), choose a sequence $\{\chi_k\}_k\in\mathbb{N}$ of cut-off functions, $\chi_k\in C_c^\infty(\mathbb{R}^n)$, $\mathbf{1}_{B_k(0)}\leq\chi_k\leq\mathbf{1}_{B_{2k}(0)}$. For $u\in C_b^2(\mathbb{R}^n)$ the sequence $u_k:=u\chi_k$ satisfies the conditions (2.10) of Lemma 2.5, therefore

$$(u_k,-p(\widetilde{x,D})u_k)=(u_k,I(p)u_k)\in\mathrm{EX}\big(-p(\widetilde{x,D})\big)$$

and, again by Lemma 2.5, and dominated convergence

$$bp\text{-}\lim_{k\to\infty}(u_k, -\widetilde{p(x,D)}u_k) = (f,g), \quad g = I(p)f.$$

Since the extended generator is *bp*-closed, $(f,g) \in \mathrm{EX}\big(-\widetilde{p(x,D)}\big)$. □

Theorem 2.6 allows us, in particular, to obtain a semimartingale decomposition of the Feller process $\{X_t\}_{t\geq 0}$ generated by some $-\widetilde{p(x,D)}$ (or A) – see Section 3 below. In order to obtain good estimates for the moments of $\{X_t\}_{t\geq 0}$, the next theorem will be quite helpful.

Theorem 2.7 *Let $(A, D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$. Assume that $\|p(\bullet,\xi)\|_\infty \leq c(1+\psi(\xi))$ for some fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$. Then*

$$\big\{u \in C_\infty(\mathbb{R}^n) \ : \ (1+\psi)\hat{u} \in L^1(\mathbb{R}^n)\big\} \subset D(\widetilde{p(x,D)})$$

for every extension $(-\widetilde{p(x,D)}, D(\widetilde{p(x,D)}))$ of $-p(x,D)$ to a Feller generator.

Proof. Let χ be a cut-off function satisfying

$$\chi \in C_c^\infty(\mathbb{R}^n), \quad \mathbf{1}_{B_1(0)} \leq \chi \leq \mathbf{1}_{B_2(0)}, \quad \chi(x) = \chi(-x).$$

The sequence $\chi_k(x) := \chi(x/k)$ converges to $\mathbf{1}$ as $k \to \infty$, and the Fourier transforms satisfy $\hat{\chi}_k(\xi) = k^n \hat{\chi}(k\xi)$.

Write $\mathscr{A} := \{u \in C_\infty(\mathbb{R}^n) \ : \ (1+\psi)\hat{u} \in L^1(\mathbb{R}^n)\}$. As a first step we claim

$$u \in \mathscr{A} \quad \text{implies} \quad u_k := (2\pi)^{-n/2}\big(\hat{\chi}_k \star u\big)\chi_k \in \mathscr{A}. \tag{2.13}$$

By the convolution theorem we get

$$\begin{aligned}\int_{\mathbb{R}^n}(1+\psi(\xi))|\hat{u}_k(\xi)|\,đ\xi &= \int_{\mathbb{R}^n}(1+\psi(\xi))|\hat{\chi}_k \star (\chi_k\hat{u})(\xi)|\,đ\xi \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+\psi(\xi))|\hat{\chi}_k(\eta)|\chi_k(\xi-\eta)|\hat{u}(\xi-\eta)|\,d\eta\,đ\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+\psi(\eta/k+\xi))|\hat{\chi}(\eta)|\chi_k(\xi)|\hat{u}(\xi)|\,đ\xi\,d\eta \\ &\leq 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+\psi(\xi)) \\ &\qquad \times (1+\psi(\eta/k))|\hat{\chi}(\eta)|\chi_k(\xi)|\hat{u}(\xi)|\,đ\xi\,d\eta\end{aligned}$$

where we used Peetre's inequality (1.5) in the last step. Since

$$1 + \psi(\eta/k) \le c_\psi(1 + \|\eta/k\|^2) \le c_\psi(1 + \|\eta\|^2)$$

and $\hat{\chi} \in \mathscr{S}(\mathbb{R}^n)$, we find

$$\|(1+\psi)\hat{u_k}\|_{L^1} \le c_{\psi,n}\|(1+\psi)\hat{u}\|_{L^1} < \infty$$

and (2.13) follows as $u_k \in C_c^\infty(\mathbb{R}^n)$.

We claim now that for $u \in \mathscr{A}$ and u_k as above

$$\hat{u_k} \to \hat{u} \qquad \text{in } L^1 \text{ and a.e.} \tag{2.14}$$

Since $u \in \mathscr{A}$, we have $\hat{u} \in L^1(\mathbb{R}^n)$. Write

$$\hat{u} - \hat{u_k} = \left(\hat{u} - (2\pi)^{-n/2}\hat{\chi}_k \star \hat{u}\right) + (2\pi)^{-n/2}\hat{\chi}_k \star \left((1-\chi_k)\hat{u}\right) \ .$$

The first member of the right-hand side converges to 0 in L^1-norm – this is a well-known mollifier technique see, e.g., Kumano-go [24, Theorem I.2.7]. The second member satisfies

$$\begin{aligned}\left\|\hat{\chi}_k \star \left((1-\chi_k)\hat{u}\right)\right\|_{L^1} &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{\chi}_k(\eta)|(1-\chi_k(\xi-\eta))|\hat{u}(\xi-\eta)|\,d\eta\,d\xi \\ &= \int_{\mathbb{R}^n}(1-\chi_k(\xi))|\hat{u}(\xi)|\,d\xi \int_{\mathbb{R}^n}|\hat{\chi}_k(\eta)|\,d\eta \\ &= \int_{\mathbb{R}^n}(1-\chi_k(\xi))|\hat{u}(\xi)|\,d\xi\|\hat{\chi}\|_{L^1} \to 0\end{aligned}$$

as $k \to \infty$ as is easily seen be dominated convergence. Passing to a subsequence, if necessary, we may assume that $\hat{u_k} \to \hat{u}$ almost everywhere.

The calculations leading to (2.13) and yet another application of dominated convergence now imply

$$\lim_{k\to\infty}\|(1+\psi)\hat{u_k}\|_{L^1} = \|(1+\psi)\hat{u}\|_{L^1}. \tag{2.15}$$

Since $(1+\psi)\hat{u_k} \in L^1(\mathbb{R}^n)$ and $(1+\psi)\hat{u} \in L^1(\mathbb{R}^n)$ such that $(1+\psi)\hat{u_k} \to (1+\psi)\hat{u}$ almost everywhere, we get even $(1+\psi)\hat{u_k} \to (1+\psi)\hat{u}$ in L^1, and thus

$$\begin{aligned}\|p(\bullet,D)u_j - p(\bullet,D)u_k\|_\infty &= \sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)(\hat{u_j}(\xi)-\hat{u_k}(\xi))\,d\!\!\!{}^-\xi\right| \\ &\le c_{\psi,n}\left\|(1+\psi)(\hat{u_j}-\hat{u_k})\right\|_{L^1} \ \to \ 0 \\ &\qquad \text{as} \quad j,k\to\infty \ .\end{aligned}$$

We have thus shown that for every $u \in \mathscr{A}$ there is a sequence $u_k \in C_c^\infty(\mathbb{R}^n) \subset D(\widetilde{p(x,D)})$ such that $u_k \to u$ uniformly (since $\hat{u_k} \to \hat{u}$ in L^1) and $\{p(\bullet,D)u_k\}_{k\in\mathbb{N}}$ is a $C_\infty(\mathbb{R}^n)$-Cauchy sequence. From the closedness of $-\widetilde{p(x,D)}$ we deduce that $u \in D(\widetilde{p(x,D)})$ and $-p(\bullet,D)u_k \to -\widetilde{p(x,D)}u$. □

Remark 2.8 (1) We may replace $\psi(\xi)$ in Theorem 2.7 by $\|p(\bullet,\xi)\|_\infty$ if this function satisfies a Peetre-type inequality. Then

$$\left\{u \in C_\infty(\mathbb{R}^n) : (1 + \|p(\bullet,\xi)\|_\infty)\hat{u} \in L^1(\mathbb{R}^n)\right\} \subset D(\widetilde{p(x,D)}) \qquad (2.16)$$

whenever $\|p(\bullet,\xi)\|_\infty$ is finite.

In general, $\xi \mapsto \|p(\bullet,\xi)\|_\infty$ will not be negative definite. However, the proof of Theorem 2.7 still works if $\|p(\bullet,\xi)\|_\infty$ satisfies a Peetre-type inequality. For fixed x and all $\xi, \eta \in \mathbb{R}^n$ we know from (1.5)

$$(1 + |p(x,\xi+\eta)|) \leq 2(1 + |p(x,\xi)|)(1 + |p(x,\eta)|)$$

since $\xi \mapsto p(x,\xi)$ is negative definite. Passing to the supremum over all $x \in \mathbb{R}^n$ yields the desired inequality.

(2) The set $\mathscr{A} := \left\{u \in C_\infty(\mathbb{R}^n) : (1 + \psi)\hat{u} \in L^1(\mathbb{R}^n)\right\}$ appearing in Theorem 2.7 fits nicely into the scale of spaces $\mathscr{B}_{p,k}$ introduced in [11, Section 10.1]. In this notation one has, in fact, $\mathscr{A} = \mathscr{B}_{1,(1+\psi)} \cap C_\infty(\mathbb{R}^n)$. This is, from a structural point of view, quite interesting. Further results in this direction can be found in the forthcoming book [18] by N. Jacob.

(3) The result of Theorem 2.7 allows to extend previous results on conservativeness, Hausdorff dimension, and Besov embeddings (cf. [31, 32, 30]) to the class of Feller processes considered in this paper.

3 The semimartingale nature of Feller processes

An n-dimensional *Feller process* $(\Omega, \mathfrak{F}, \mathbb{P}^x, x \in \mathbb{R}^n, \{X_t\}_{t\geq 0}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{R}^n, \mathfrak{B})$ is a strong Markov process with the additional properties that $x \mapsto \mathbb{E}^x(u(X_t))$ is in $C_\infty(\mathbb{R}^n)$ and $\lim_{t\to 0} \mathbb{E}^x(u(X_t)) = u(x)$ whenever $u \in C_\infty(\mathbb{R}^n)$. It is well known that there is a correspondence between Feller semigroups $\{T_t\}_{t\geq 0}$ and Feller processes $\{X_t\}_{t\geq 0}$ which is expressed by

$$T_t u(x) = \mathbb{E}^x(u(X_t)), \quad u \in C_b(\mathbb{R}^n), \quad t \geq 0, \quad x \in \mathbb{R}^n \ . \qquad (3.1)$$

In particular, T_t, originally defined on $C_\infty(\mathbb{R}^n)$, extends onto $C_b(\mathbb{R}^n)$ and $B_b(\mathbb{R}^n)$.

In this section we want to study the structure of Feller processes. As before, we assume that the generator of $\{T_t\}_{t\geq 0}$ be some extension $-\widetilde{p(x,D)}$ of the pseudo differential operator $-p(x,D)$ with symbol $-p(x,\xi)$ given by (1.3). Our main objective is to show that $\{X_t\}_{t\geq 0}$ is a semimartingale and that its semimartingale characteristics (in the sense of [21]) are determined by the Lévy characteristics of the symbol.

We will frequently use the notion of (*pre-*)*stopping* of a process. Let τ be some stopping time for $\{X_t\}_{t\geq 0}$. Then the (pre-)stopped processes X_t^τ and $X_t^{\tau-}$ are given by

$$X_t^\tau(\omega) := X_{t\wedge\tau(\omega)}(\omega) \quad \text{and}$$
$$X_t^{\tau-}(\omega) := X_t(\omega)\mathbf{1}_{[0,\tau(\omega))}(t) + X_{\tau(\omega)-}(\omega)\mathbf{1}_{[\tau(\omega),\infty)}(t) \ .$$

The following Lemma is taken from Protter [26, Theorem II.6]. It shows essentially that a prelocal semimartingale is already a semimartingale.

Lemma 3.1 *Let $\{X_t\}_{t\geq 0}$ be a càdlàg adapted process with values in $\mathbb{R}$, $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of positive random times which increase a.s. to ∞, and let $\{X_{k,t}\}_{t\geq 0}, k \in \mathbb{N}$, be a sequence of semimartingales such that $X_\bullet^{\tau_k -} = X_{k,\bullet}^{\tau_k -}$ for all $k \in \mathbb{N}$. Then $\{X_t\}_{t\geq 0}$ is a semimartingale.*

Rather than using the more appropriate notion *vector of semimartingales*, we will call an n-dimensional process *semimartingale*, if its component processes are semimartingales. Note that this is in line with the notation of [21]. *Without further mentioning we will understand the semimartingale property always w.r.t. the family $\{\mathbb{P}^x\}_{x\in\mathbb{R}^n}$*. In order to simplify things, we will assume that the life-time ζ of the process is a.s. infinite. Otherwise, the statements below remain valid, but on the set $\{\omega : \zeta(\omega) < \infty\}$ only. A necessary and sufficient condition for $\zeta = \infty$ (a.s.) is that $p(x,0) \equiv 0$, cf. the discussion at the beginning of Section 4.

Lemma 3.2 *Let $(A, D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$* and *$A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$ with symbol $-p(x,\xi)$ given by* (1.3), *$\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$, and $p(x,0)\equiv 0$. Every* (*n-dimensional*) *Feller process $\{X_t\}_{t\geq 0}$ generated by some extension $-\widetilde{p(x,D)}$ of $-p(x,D)$ to a Feller generator is a semimartingale* (*under any $\mathbb{P}^x$, $x\in\mathbb{R}^n$*).

Proof. It is well known that for any càdlàg Feller process and all u in the domain of $-\widetilde{p(x,D)}$ the process

$$M_t := u(X_t) - u(X_0) + \int_0^t \widetilde{p(x,D)}u(x)\big|_{x=X_s}\,ds, \quad t\geq 0 \ , \tag{3.2}$$

is a martingale. Pick for $j = 1,\dots,n$ sequences $\phi_k^j \in C_c^\infty(\mathbb{R}^n)$ such that

$$\phi_k^j|B_k(0) = x_j, \quad \phi_k^j|B_{2k}^c(0) = 0, \quad \text{and} \quad \|\phi_k^j\|_\infty \leq k+1 \ .$$

Clearly, $\phi_k^j(\bullet - x) \in D(p(x,D)) \subset D(\widetilde{p(x,D)})$, and under $\mathbb{P}^x$

$$M_{k,t}^{(j)} := \phi_k^j(X_t - x) + \int_0^t p(y, D_y)\phi_k^j(y-x)\big|_{y=X_s}\,ds$$

are martingales for every $k \in \mathbb{N}$ and $j = 1, \dots, n$. Since

$$|M_{k,t}^{(j)}| \le \|\phi_k^j\|_\infty + t\|p(\bullet, D)\phi_k^j\|_\infty < \infty \ ,$$

$M_{k,\bullet}^{(j)}$ is even an L^2_{loc}-martingale. The above inequality shows also that the integral term in the decomposition of $M_{k,\bullet}^{(j)}$ is a process of bounded variation in finite time intervals. By the canonical decomposition of semimartingales (cf. e.g. [26, Theorem III.1]) we conclude that the processes

$$X_{k,t}^{(j)} := \phi_k^j(X_t - x) = M_{k,t}^{(j)} - \int_0^t \big(p(\bullet, D)\phi_k^j(\bullet - x)\big)(X_s)\,ds$$

are semimartingales.

Set $\tau_k := \tau_k^x := \inf\{t \ge 0 : \|X_t - x\| > k\}$. Then τ_k are stopping times converging a.s. to ∞. By construction, we have

$$\big(X_{k,\bullet}^{(j)}\big)^{\tau_k-} = \big(\phi_k^j(X_\bullet - x)\big)^{\tau_k-} = \big(X_\bullet^{(j)} - x_j\big)^{\tau_k-} \ .$$

For $t < \tau_k$ this equality is straightforward, if $t \ge \tau_k$ it is easily seen from

$$\begin{aligned}\big(\phi_k^j(X_t - x)\big)^{\tau_k-} &= \phi_k^j(X_{\tau_k-} - x)\\ &= \lim_{r<\tau_k, r\to\tau_k} \phi_k^j(X_r - x)\\ &= \lim_{r<\tau_k, r\to\tau_k} (X_r^{(j)} - x_j) = X_{\tau_k-}^{(j)} - x_j \ .\end{aligned}$$

Now Lemma 3.1 applies and shows that $\{X_t\}_{t\ge0}$ is a semimartingale. $\square$

Let us briefly recall the notion of *characteristics of a semimartingale*, see [21]. Let $\chi = \chi_R \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $\mathbf{1}_{B_R(0)} \le \chi_R \le \mathbf{1}_{B_{2R}(0)}$, $R > 0$ fixed. Set

$$J_t(\chi) := \sum_{s\le t}(1 - \chi(\Delta X_s))\Delta X_s, \qquad \check{X}_t(\chi) := X_t - J_t(\chi) \tag{3.3}$$

and observe that $\{\check{X}_t(\chi)\}_{t\ge0}$ is a semimartingale with bounded jumps, hence a special semimartingale admitting the unique canonical decomposition

$$\check{X}_t(\chi) = X_0 + M_t(\chi) + B_t(\chi) \tag{3.4}$$

where $M_t(\chi)$ is a local martingale and $B_t(\chi)$ is a previsible process with paths of finite variation on compact intervals.

Definition 3.3 *Let* $\{X_t\}_{t\geq 0}$, $X_t = (X_t^{(1)}, \dots, X_t^{(n)})$, *be a semimartingale and* $\chi = \chi_R$ *be as above. Moreover, let* $B_t = B_t(\chi)$ *be the previsible process appearing in* (3.4), X_t^c *be the continuous martingale part of* X_t, *and* $v(\omega, ds, dy)$ *the compensator* (*i.e. dual predictable projection*) *of the jump measure*

$$\mu^X(\omega, ds, dy) := \sum_{s:\Delta X_s \neq 0} \delta_{(s,\Delta X(s,\omega))}(ds, dy)$$

of the process $\{X_t\}_{t\geq 0}$. *Denote by* $C_\bullet = (C_\bullet^{jk})_{j,k=1}^n$ *the quadratic co-variation matrix* $C_t^{jk} := \langle X^{(j),c}, X^{(k),c}\rangle_t$ *for* $\{X_t^c\}_{t\geq 0}$.

Then (B, C, v) *is called (a version of) the characteristics of the semimartingale* $\{X_t\}_{t\geq 0}$ (*relative to* χ).

It is obvious from the definition of characteristics that $(B, C, \chi \cdot v)$ is a version of the characteristics of $\{\check{X}_t\}_{t\geq 0}$ of (3.3), (3.4).

One could equivalently define (B, C, v) (relative to χ) by the requirement that for all $f \in C_b^2(\mathbb{R}^n)$ the process

$$\begin{aligned} M_t^f := f(X_t) - f(X_0) - \int_0^t \sum_{j=1}^n \partial_j f(X_{s-})\, dB_s^{(j)} \\ - \frac{1}{2}\int_0^t \sum_{j,k=1}^n \partial_j\partial_k f(X_{s-})\, dC_s^{jk} \\ - \int_0^t \int_{\mathbb{R}^n} (f(X_{s-} + y) - f(X_{s-}) - \chi(y) y \cdot \nabla f(X_{s-})) v(\bullet, ds, dy) \end{aligned} \tag{3.5}$$

is a local martingale, cf. [21, Theorem II.2.42].

The characteristics admit the following *canonical representation* of the semimartingale $\{X_t\}_{t\geq 0}$,

$$\begin{aligned} X_t &= X_0 + X_t^c + \int_0^t \chi(y) y \big(\mu^X(\bullet, ds, dy) - v(\bullet, ds, dy)\big) + J_t(\chi) + B_t(\chi) \\ &= \check{X}_t(\chi) + J_t(\chi) \ . \end{aligned} \tag{3.6}$$

The integrals are to be understood componentwise.

The next definition is again taken from [21]

Definition 3.4 *A semimartingale* $\{X_t\}_{t\geq 0}$ *is called homogeneous diffusion with jumps if its characteristics have the form*

$$B_t^{(j)}(\omega) = \int_0^t b^{(j)}(X_s(\omega))\, ds \ ,$$

$$C_t^{jk}(\omega) = \int_0^t c^{jk}(X_s(\omega))\,ds \ ,$$
$$\nu(\omega,\, ds, dy) = N(X_s(\omega), dy)\,ds \ ,$$

where $b^{(j)}, c^{jk} : \mathbb{R}^n \to \mathbb{R}$ *are measurable functions,* $c = (c^{jk})_{j,k=1}^n$ *is a positive semidefinite matrix, and* $N(x, \bullet)$ *is a Borel transition kernel such that* $N(x, \{0\}) = 0$.

The notion *diffusion* process with jumps can be a bit misleading: every Lévy process is a diffusion with jumps, even a pure jump process without continuous martingale part (e.g. a symmetric α-stable process) can be a "diffusion" with jumps.

Theorem 3.5 *Let* $(A, D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^n) \subset D(A)$ *and* $A|C_c^\infty(\mathbb{R}^n) = -p(x, D)$ *with symbol* $-p(x,\xi)$ *given by* (1.3), $\|p(\bullet,\xi)\|_\infty \leq c(1 + \|\xi\|^2)$, *and* $p(x, 0) \equiv 0$. *Denote by* $\{X_t\}_{t\geq 0}$ *the Feller process generated by any extension* $\widetilde{-p(x,D)}$ *of* $-p(x,D)$ *to a Feller generator. Then* $\{X_t\}_{t\geq 0}$ *is a homogeneous diffusion with jumps and its semimartingale characteristics* (B, C, ν) *with respect to* $\chi = \chi_R$ *are determined by the Lévy characteristics* $(\ell(x), Q(x), N(x, dy))$ *of* $p(x,\xi)$:

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell_j(X_s(\omega))\,ds \\ &\quad + \int_0^t \int_{y\neq 0} \left(\frac{y_j}{1 + \|y\|^2} - \chi_R(y) y_j \right) N(X_s(\omega), dy)\,ds \\ C_t^{jk}(\omega) &= 2\int_0^t q_{jk}(X_s(\omega))\,ds \\ \nu(\omega, ds, dy) &= N(X_s(\omega), -dy)\,ds \end{aligned}$$

where $\chi = \chi_R$ *is the cut-off function introduced in Definition* 3.3.

Proof. Since for suitable constants

$$|\chi_R(y) y_j| \leq c_j \frac{|y_j|}{1 + \|y\|^2} \quad \text{and} \quad \left| \frac{y_j \xi_j}{1 + \|y\|^2} - \chi_R(y) y_j \xi_j \right| \leq C_j \frac{|y_j| \cdot |\xi_j|}{1 + \|y\|^2} \ ,$$

(B, C, ν) from above are well-defined. By Lemma 3.1, $\{X_t\}_{t\geq 0}$ is a semimartingale, and it remains to check that (3.5) is a local martingale for all $f \in C_b^2(\mathbb{R}^n)$. From Theorem 2.6 we know that $(f, I(p)f)$ is in the extended generator $\mathrm{EX}\big(\widetilde{-p(x,D)}\big)$, thus – cf. [7, Proposition IV.1.7] – the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t I(p)f(X_{s-})\,ds$$

is a martingale. Using the explicit representation (2.7) of $I(p)$, we may rewrite M_t^f in the following form

$$\begin{aligned} M_t^f = f(X_t) - f(X_0) - \int_0^t \Bigg[\ell(X_s). \\ + \int_{y\neq 0}\left(\frac{y}{1+\|y\|^2} - \chi(y)y\right)N(X_s,dy)\Bigg]\cdot\nabla f(X_{s-})\,ds \\ - \int_0^t \Bigg(\sum_{j,k=1}^n q_{jk}(X_s)\partial_j\partial_k f(X_{s-}) \\ + \int_{y\neq 0}(f(X_{s-}-y) - f(X_{s-}) + \chi(y)y\cdot\nabla f(X_{s-}))N(X_s,dy)\Bigg)\,ds \end{aligned}$$

which is but (3.5). □

Let us, for further reference purposes, emphasize the following observation made in the proof of Theorem 3.5.

Corollary 3.6 *Let $(A, D(A))$, $-p(x,D)$, $\{X_t\}_{t\geq 0}$ be as in the preceding Theorem. Then the process*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t I(p)f(X_{s-})\,ds$$

is for every $f \in C_b^2(\mathbb{R}^n)$ an L^2_{loc}-martingale.

Proof. The martingale property of M_t^f has already been established in the proof of Theorem 3.5. Since the estimate (2.9) carries over to $I(p)$, we have that M_t^f is a bounded random variable for t varying in compact sets, hence square-integrable. □

4 Asymptotic behaviour of X_t for $t \to 0$ and $t \to \infty$

Let $\{X_t\}_{t\geq 0}$ denote an n-dimensional Feller process as in the preceding section. We are interested in the limiting behaviour of X_t under $\mathbb{P}^x$, i.e., for which $\lambda > 0$

$$\liminf_t t^{-1/\lambda} \sup_{s\leq t}\|X_s - x\| \quad \text{or} \quad \limsup_t t^{-1/\lambda}\sup_{s\leq t}\|X_s - x\|$$

are finite or infinite as $t \to 0$, $t \to \infty$. We will determine bounds for λ in terms of the symbol $-p(x,\xi)$ of the infinitesimal generator. For Lévy processes, i.e., in the constant-coefficient case $p(x,\xi) = \psi(\xi)$, similar criteria were given by Pruitt [27]. Our exposition is heavily influenced by this paper; we adopted, in particular, the idea to use Lemma 4.1 which has its analogue in [27, p. 949, Lemma]. To our knowledge, the only studies for non-constant coefficients are the papers [25, 22] by Negoro and Negoro & Kikuchi, containing some short-time asymptotics for stable-like processes. We should like to mention that these results are encompassed by our theorems, see the examples at the end of Section 5.

When studying the long-term behaviour we will have to assume that $\{X_t\}_{t\geq 0}$ has *a.s. infinite life-time*. The semigroup equivalent would be to assume that the associated Feller semigroup $\{T_t\}_{t\geq 0}$ is *conservative*. A sufficient (and in fact also necessary, cf. [33]) condition for this is that the symbol $-p(x,\xi)$ satisfies $p(x,0) \equiv 0$, see [31, 33], that is to say that

$$p(x,\xi) = -\mathrm{i}\ell(x)\cdot\xi + \xi\cdot Q(x)\xi + \int_{y\neq 0}\left(1 - \mathrm{e}^{-\mathrm{i}y\cdot\xi} - \frac{\mathrm{i}y\cdot\xi}{1+\|y\|^2}\right)N(x,dy) \ . \tag{4.1}$$

The key ingredient for our investigations are estimates of the distribution function of the *maximal process*,

$$(X_\bullet - x)_t^* := \sup_{s\leq t}\|X_s - x\| \ . \tag{4.2}$$

For Lévy processes the following Lemma is due to W. Pruitt [27]. In our situation the proof is much more involved and we will postpone it until Section 6.

Lemma 4.1 *Let* $(A, D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^n) \subset D(A)$ *and* $A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$. *Assume that the symbol* $-p(x,\xi)$ *is given by* (4.1), *satisfies* $\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$, *and that* $\{X_t\}_{t\geq 0}$ *is the Feller process generated by some extension of* $-p(x,D)$ *to a Feller generator. Then the following estimates hold true*:

$$\mathbb{P}^x\big((X_\bullet - x)_t^* \geq R\big) \leq c_n t H(x,R), \quad t \geq 0, \quad R > 0 \ , \tag{4.3}$$

where c_n *is a constant depending only on the space dimension*, *and*

$$H(x,R) := \sup_{\|y-x\|\leq 2R}\ \sup_{\|\epsilon\|\leq 1}\left(\int_{-\infty}^{\infty} \mathrm{Re}\ p\Big(y,\frac{\rho\epsilon}{R}\Big)g(\rho)\,d\rho + \Big|p\Big(y,\frac{\epsilon}{R}\Big)\Big|\right) \ .$$

(*The density* $g(\rho) = g_1(\rho)$ *is given by* (2.5), (2.6).)

If, in addition, $|\operatorname{Im} p(x,\xi)| \le c_0 \operatorname{Re} p(x,\xi)$ *for some absolute constant* $c_0 \ge 1$, *then*

$$\mathbb{P}^x\big((X_\bullet - x)_t^* < R\big) \le \frac{c_\kappa}{t h(x,R)}, \quad t \ge 0, \quad R > 0, \tag{4.4}$$

$$\mathbb{P}^x\big((X_\bullet - x)_t^* < R/2\big) \le \frac{c_\kappa^2}{t^2 h^2(x,R)}, \quad t \ge 0, \quad R > 0, \tag{4.5}$$

where the constant c_κ *depends only on* $\kappa := \big(4 \arctan\big(\frac{1}{2c_0}\big)\big)^{-1}$, *and*

$$h(x,R) := \inf_{\|y-x\| \le 2R} \sup_{\|\epsilon\| \le 1} \operatorname{Re} p\Big(y, \frac{\epsilon}{4\kappa R}\Big) .$$

The growth of the sample paths at $t = \infty$ is governed by the following *indices* which are generalizations of the Blumenthal-Getoor index β and the Pruitt index δ for Lévy processes, cf. [4, 27]. Recently, indices of this kind have attracted renewed interest, see e.g. the paper [28].

Definition 4.2 *Let* $h(x,R)$ *and* $H(x,R)$ *be as in Lemma* 4.1, *and set*

$$h(R) := \inf_{x \in \mathbb{R}^n} h(x,R) \quad \textit{and} \quad H(R) := \sup_{x \in \mathbb{R}^n} H(x,R) .$$

Then the quantities

$$\beta_0 := \sup\Big\{\lambda \ge 0 : \limsup_{R\to\infty} R^\lambda H(R) = 0\Big\} ,$$

$$\underline{\beta}_0 := \sup\Big\{\lambda \ge 0 : \liminf_{R\to\infty} R^\lambda H(R) = 0\Big\} ,$$

$$\bar{\delta}_0 := \sup\Big\{\lambda \ge 0 : \limsup_{R\to\infty} R^\lambda h(R) = 0\Big\} ,$$

$$\delta_0 := \sup\Big\{\lambda \ge 0 : \liminf_{R\to\infty} R^\lambda h(R) = 0\Big\} ,$$

are called indices of $\{X_t\}_{t\ge 0}$ *(or* $p(x,\xi)$*) at the origin.*

From the estimates

$$\begin{aligned}|p(x,\xi)| \ge \operatorname{Re} p(x,\xi) &\ge \int_{|y\cdot\xi|\le 1} (1-\cos y\cdot\xi) N(x,dy) \\ &\ge c \int_{|y\cdot\xi|\le 1} |y\cdot\xi|^2 N(x,dy)\end{aligned}$$

it becomes immediately clear that all of the above indices are well-defined and that $\beta_0, \underline{\beta}_0, \bar{\delta}_0, \delta_0 \in [0,2]$. A moment's thought shows

$$\beta_0 \le \underline{\beta}_0 \le \delta_0 \quad \text{and} \quad \beta_0 \le \bar{\delta}_0 \le \delta_0 \ .$$

In Section 5 below we will give alternative characterizations of these indices and explain their relation with the *classical* indices of Blumenthal and Getoor for Lévy processes.

Theorem 4.3 *Let $(A, D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n) = -p(x, D)$. Assume that the symbol $-p(x,\xi)$ is given by* (4.1) *and satisfies $\|p(\bullet,\xi)\|_\infty \le c(1+\|\xi\|^2)$ and* $|\ \mathrm{Im}\ p(x,\xi)| \le c_0\ \mathrm{Re}\ p(x,\xi)$. *Then the Feller process $\{X_t\}_{t\ge 0}$ generated by any extension of $-p(x,D)$ satisfies*

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\bullet - x)_t^* = 0 \quad a.s. \quad (\mathbb{P}^x) \quad for\ all \quad \lambda < \beta_0 \tag{4.6}$$

$$\liminf_{t\to\infty} t^{-1/\lambda}(X_\bullet - x)_t^* = 0 \quad a.s. \quad (\mathbb{P}^x) \quad for\ all \quad \beta_0 \le \lambda < \underline{\beta}_0 \tag{4.7}$$

$$\limsup_{t\to\infty} t^{-1/\lambda}(X_\bullet - x)_t^* = \infty \quad a.s. \quad (\mathbb{P}^x) \quad for\ all \quad \bar{\delta}_0 < \lambda \le \delta_0 \tag{4.8}$$

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\bullet - x)_t^* = \infty \quad a.s. \quad (\mathbb{P}^x) \quad for\ all \quad \lambda > \delta_0 \tag{4.9}$$

Proof. In order to prove (4.6) we assume that $\lambda < \beta_0$ and pick $\lambda < \alpha_2 < \alpha_1 < \beta_0$. By (4.3) we see

$$\begin{aligned}&\mathbb{P}^x\big((X_\bullet - x)_t^* \ge t^{1/\alpha_2}\big)\\ &\quad \le c_n t H(x, t^{1/\alpha_2}) \le c_n t H(t^{1/\alpha_2}) \le c_n' t \big(t^{1/\alpha_2}\big)^{-\alpha_1} = c_n' t^{1-\alpha_1/\alpha_2}\end{aligned}$$

where we only used the definition of β_0 and the fact that $\alpha_1 < \beta_0$. For the sequence $t_k := 2^k$, $k \in \mathbb{N}$, we obtain

$$\sum_{k=1}^\infty \mathbb{P}^x\big((X_\bullet - x)_{t_k}^* \ge t_k^{1/\alpha_2}\big) \le c_n \sum_{k=1}^\infty 2^{k(1-\alpha_1/\alpha_2)} < \infty \ ,$$

and the Borel-Cantelli Lemma shows that $\mathbb{P}^x(\limsup_{k\to\infty} \{(X_\bullet - x)_{t_k}^* \ge t_k^{1/\alpha_2}\}) = 0$, hence $(X_\bullet - x)_{t_k}^* \le t_k^{1/\alpha_2}$ for "eventually all" $k \in \mathbb{N}$. Choose $t \in [t_k, t_{k+1})$, and observe that for every ω and sufficiently large k (depending on ω)

$$(X_\bullet(\omega) - x)_t^* \le (X_\bullet(\omega) - x)_{t_{k+1}}^* \le t_{k+1}^{1/\alpha_2} \le 2^{1/\alpha_2} t^{1/\alpha_2} \ .$$

Therefore, for $\lambda < \alpha_2$

$$t^{-1/\lambda}(X_\bullet - x)_t^* \le 2^{1/\alpha_2} t^{1/\alpha_2 - 1/\lambda} \to 0 \quad \text{a.s.}\ (\mathbb{P}^x)\ \text{as}\ t \to \infty \ .$$

The proof of (4.9) is similar: choose $\delta_0 < \alpha_1 < \alpha_2 < \lambda$ and observe that (4.4) and the definition of δ_0 imply

$$\mathbb{P}^x\big((X_\bullet - x)_t^* < t^{1/\alpha_2}\big) \le \frac{c_\kappa}{th(x, t^{1/\alpha_2})} \le \frac{c_\kappa}{th(t^{1/\alpha_2})} \le c'_\kappa t^{-1}\big(t^{1/\alpha_2}\big)^{\alpha_1} \ .$$

Again we take $t_k := 2^k$, $k \in \mathbb{N}$, and find

$$\sum_{k=1}^{\infty} \mathbb{P}^x\big((X_\bullet - x)_{t_k}^* < t_k^{1/\alpha_2}\big) \le c'_\kappa \sum_{k=1}^{\infty} 2^{k(\alpha_1/\alpha_2 - 1)} < \infty \ ,$$

and by the Borel-Cantelli Lemma we deduce that $(X_\bullet - x)_{t_k}^* \ge t_k^{1/\alpha_2}$ for "eventually all" k. If k is sufficiently large and $t \in [t_k, t_{k+1})$

$$(X_\bullet(\omega) - x)_t^* \ge (X_\bullet(\omega) - x)_{t_k}^* \ge t_k^{1/\alpha_2} \ge 2^{-1/\alpha_2} t^{1/\alpha_2} \ ,$$

for every ω, and, since $\lambda > \alpha_2$,

$$t^{-1/\lambda}(X_\bullet - x)_t^* \ge 2^{-1/\alpha_2} t^{1/\alpha_2 - 1/\lambda} \to \infty \quad \text{a.s. } (\mathbb{P}^x) \text{ as } t \to \infty \ .$$

For the proof of (4.7) and (4.8) we assume that $\lambda > \sigma > \overline{\delta}_0$ and $\mu < \tau < \underline{\beta}_0$ and choose sequences such that

$$\lim_{k\to\infty} t_k^\tau H(t_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} s_k^\sigma h(s_k) = \infty.$$

By Lemma 4.1 (4.4) we see

$$\mathbb{P}^x\big((X_\bullet - x)_{s_k^\sigma}^* < s_k\big) \le \frac{c}{s_k^\sigma h(s_k)} \to 0 \quad \text{as} \quad k \to \infty \ ,$$

and Fatou's Lemma implies

$$\begin{aligned} 0 &= \liminf_{k\to\infty} \mathbb{P}^x\big((X_\bullet - x)_{s_k^\sigma}^* < s_k\big) \\ &= 1 - \limsup_{k\to\infty} \mathbb{P}^x\big((X_\bullet - x)_{s_k^\sigma}^* \ge s_k\big) \\ &\ge 1 - \mathbb{P}^x\Big(\limsup_{k\to\infty} \big\{(X_\bullet - x)_{s_k^\sigma}^* \ge s_k\big\}\Big) \end{aligned}$$

which is but to say that

$$\mathbb{P}^x\big((X_\bullet - x)_{s_k^\sigma}^* \ge s_k \text{ i.o.}\big) = \mathbb{P}^x\big(s_k^{-1}(X_\bullet - x)_{s_k^\sigma}^* \ge 1 \text{ i.o.}\big) = 1$$

– "i.o." stands for "infinitely often". Hence,

$$\limsup_{t\to\infty} t^{-1/\sigma}(X_\bullet - x)_t^* \ge 1 \quad \text{for} \quad \sigma < \lambda \ ,$$

and (4.8) follows. The proof of (4.7) is similar, with s_k, σ replaced by t_k, τ. □

In the proof of Theorem 4.3 we did not always need the condition $|\operatorname{Im} p(x,\xi)| \le c_0 \operatorname{Re} p(x,\xi)$. An important special case is covered by the next Corollary.

Corollary 4.4 *Let* $(A, D(A))$, $-p(x, D)$, *and* $\{X_t\}_{t\geq 0}$ *be as in Theorem* 4.3, *but assume that the symbol* $-p(x,\xi)$ *given by* (4.1) *satisfies only* $\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$. *Then* (4.6) *and* (4.7) *remain valid and one has, in particular*,

$$\sup_{s\leq t}\|X_s(\omega)-x\| \leq c_\omega(1+t)^{1/\lambda} \qquad a.s.\ (\mathbb{P}^x)\ \textit{for all}\ \lambda < \beta_0 \tag{4.10}$$

with an a.s. finite constant c_ω, $\mathbb{P}^x(c_\bullet < \infty) = 1$.

We will now investigate the behaviour as $t \to 0$. Other than for the long-time asymptotics, one should expect a strong dependence on the initial value $X_0 = x$ of the process.

Definition 4.5 *Let* $h(x,R)$ *and* $H(x,R)$ *be the functions introduced in Lemma* 4.1. *The quantities*

$$\beta^x_\infty := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda H(x,R) = 0\right\}$$

$$\underline{\beta}^x_\infty := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda H(x,R) = 0\right\}$$

$$\overline{\delta}^x_\infty := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda h(x,R) = 0\right\}$$

$$\delta^x_\infty := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda h(x,R) = 0\right\}$$

are called indices of $\{X_t\}_{t\geq 0}$ (*or* $p(x,\xi)$) *at infinity.*

Clearly, $\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$ guarantees $\beta^x_\infty, \underline{\beta}^x_\infty, \overline{\delta}^x_\infty, \delta^x_\infty \in [0,2]$, and one has

$$\delta^x_\infty \leq \underline{\beta}^x_\infty \leq \beta^x_\infty \quad \text{and} \quad \delta^x_\infty \leq \overline{\delta}^x_\infty \leq \beta^x_\infty \ .$$

For a thorough discussion and concrete examples of these indices we refer to Section 5.

Since the proof of the next Theorem parallels Theorem 4.3 (using the indices at infinity of Definition 4.5 instead of the indices at zero of Definition 4.2), it is not necessary to provide details of the proof.

Theorem 4.6 *Let* $(A, D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^n) \subset D(A)$ *and* $A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$. *Assume that the symbol* $-p(x,\xi)$ *is given by* (4.1) *and satisfies* $\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$ *and* $|\mathrm{Im}\, p(x,\xi)| \leq c_0\ \mathrm{Re}\, p(x,\xi)$. *Then the Feller process* $\{X_t\}_{t\geq 0}$ *generated by any extension of* $-p(x,D)$ *satisfies at every starting point* $x \in \mathbb{R}^n$

$$\lim_{t\to 0} t^{-1/\lambda}(X_\bullet - x)^*_t = 0 \quad a.s. \quad (\mathbb{P}^x) \quad \textit{for all} \quad \lambda > \beta^x_\infty \tag{4.11}$$

$$\liminf_{t\to 0} t^{-1/\lambda}(X_\bullet - x)_t^* = 0 \quad a.s. \quad (\mathbb{P}^x) \quad \textit{for all} \quad \beta_\infty^x \geq \lambda > \underline{\beta}_\infty^x \tag{4.12}$$

$$\limsup_{t\to 0} t^{-1/\lambda}(X_\bullet - x)_t^* = \infty \quad a.s. \quad (\mathbb{P}^x) \quad \textit{for all} \quad \overline{\delta}_\infty^x > \lambda \geq \delta_\infty^x \tag{4.13}$$

$$\lim_{t\to 0} t^{-1/\lambda}(X_\bullet - x)_t^* = \infty \quad a.s. \quad (\mathbb{P}^x) \quad \textit{for all} \quad \lambda < \delta_\infty^x \tag{4.14}$$

We will conclude this section with another application of Lemma 4.1, the calculation of the expected first passage times $\sigma_R := \sigma_R^x := \inf\{t > 0 : \|X_t - x\| > R\}$.

Theorem 4.7 *Let $(A, D(A))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^n) \subset D(A)$ and $A|C_c^\infty(\mathbb{R}^n) = -p(x,D)$. Assume that the symbol $-p(x,\xi)$ is given by* (4.1) *and satisfies $\|p(\bullet,\xi)\|_\infty \leq c(1 + \|\xi\|^2)$ and $|\mathrm{Im}\, p(x,\xi)| \leq c_0 \mathrm{Re}\, p(x,\xi)$. Then the first passage time σ_R of the Feller process $\{X_t\}_{t\geq 0}$ generated by any extension of $-p(x,D)$ satisfies*

$$\frac{c}{H(x,R)} \leq \mathbb{E}^x(\sigma_R) \leq \frac{C}{h(x,R)} \tag{4.15}$$

with absolute constants c, C depending only on the space dimension n and c_0.

Proof. Observe that

$$\left\{(X_\bullet - x)_t^* < R\right\} \subset \left\{\sigma_R > t\right\} \subset \left\{(X_\bullet - x)_t^* \leq R\right\} . \tag{4.16}$$

holds. Using (4.5) we find for any $\xi > 0$

$$\begin{aligned} \mathbb{E}^x(\sigma_R) &= \int_0^\infty \mathbb{P}^x(\sigma_R > t)\, dt \leq \xi + \int_\xi^\infty \mathbb{P}^x\big((X_\bullet - x)_t^* < 2R\big)\, dt \\ &\leq \xi + \frac{c_\kappa^2}{h^2(x,4R)} \int_\xi^\infty \frac{dt}{t^2} = \xi + \frac{c_\kappa^2}{h^2(x,4R)} \frac{1}{\xi} \quad . \end{aligned}$$

Minimizing over $\xi > 0$ gives an upper bound in (4.15) with $C = 2c_\kappa$ and $h(x, 4R)$. The assertion follows using the subadditivity of $\xi \mapsto \sqrt{p(x,\xi)}$ with $C = 32c_\kappa$. Similarly, by (4.16) and (4.3) we get for any $\xi > 0$

$$\begin{aligned} \mathbb{E}^x(\sigma_R) &= \int_0^\infty \mathbb{P}^x(\sigma_R > t)\, dt \geq \int_0^\xi \mathbb{P}^x\big((X_\bullet - x)_t^* < R\big)\, dt \\ &= \xi - \int_0^\xi \mathbb{P}^x\big((X_\bullet - x)_t^* \geq R\big)\, dt \geq \xi - c_n H(x,R) \int_0^\xi t\, dt \end{aligned}$$

Maximizing this expression in ξ gives the lower bound in (4.15) with $c = 1/(2c_n)$. $\square$

Remark 4.8 Combining Theorem 4.7 with Lemma 5.1 below easily shows the somewhat sharper inequality

$$\frac{c_n}{\sup\limits_{\substack{\|x-y\|\le 2R\\ \|\epsilon\|\le 1}} \operatorname{Re} p\Big(x, \frac{\epsilon}{R}\Big)} \le \mathbb{E}^x(\sigma_R) \le \frac{C_\kappa}{\inf\limits_{\|y-x\|\le 2R} \sup\limits_{\|\epsilon\|\le 1} \operatorname{Re} p\Big(y, \frac{\epsilon}{4\kappa R}\Big)} \tag{4.17}$$

with $\kappa = \kappa(c_0)$ from Lemma 3.1. Note that only c_n depends on the space dimension.

5 Applications, examples, and characterizations of the indices

The indices introduced in Definition 4.2 and 4.5 are modelled on their constant-coefficient counterparts, the Blumenthal-Getoor index β and the Pruitt index δ, see [4, 27]. It is not hard to see (using Lemma 5.1 below) that, if $p(x,\xi) = \psi(\xi)$,

$$\beta_0 = \overline{\delta}_0 \quad \text{and} \quad \underline{\beta}_0 = \delta_0$$

and, since there is no x-dependence,

$$\beta_\infty := \beta^x_\infty = \overline{\delta}^x_\infty \quad \text{and} \quad \delta_\infty := \underline{\beta}^x_\infty = \delta^x_\infty \ .$$

In fact, a little more holds true (as will be clear from the Lemmata below): β_∞ equals Blumenthal-and-Getoor's β,

$$\beta := \inf\left\{ \lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\psi(\xi)}{\|\xi\|^\lambda} = 0 \right\} \ ,$$

and for β_0 one has the analoguous formula

$$\beta_0 = \sup\left\{ \lambda \ge 0 : \lim_{\|\xi\|\to 0} \frac{\psi(\xi)}{\|\xi\|^\lambda} = 0 \right\}$$

The indices δ_∞, δ_0 are just Pruitt's δ (in the respective settings).

In order to give similar characterizations for the general, i.e., variable coefficient case, we need an auxiliary result.

Lemma 5.1 *Let* $\xi \mapsto p(x,\xi)$ *be for every* $x \in \mathbb{R}^n$ *be a negative definite function given by* (4.1), *and assume that* $g(\rho) = g_1(\rho)$ *is the density introduced in* (2.5), (2.6). *Then*

$$\sup_{\|\epsilon\|\le 1} \int_{-\infty}^{\infty} \operatorname{Re} p\Big(x, \frac{\rho\epsilon}{R}\Big) g(\rho)\, d\rho \le c \sup_{\|\epsilon\|\le 1} \operatorname{Re} p\Big(x, \frac{\epsilon}{R}\Big) \ , \tag{5.1}$$

with a constant $c > 0$ *depending on the density* $g(\rho)$.

Proof. For every $\epsilon \in \mathbb{R}^n$ such that $\|\epsilon\| \le 1$

$$\int_{-\infty}^{\infty} \operatorname{Re} p\Big(x, \frac{\rho\epsilon}{R}\Big) g(\rho)\, d\rho \le \sup_{\|\epsilon\|\le 1} \operatorname{Re} p\Big(x, \frac{\epsilon}{R}\Big) \int_{-1}^{1} g(\rho)\, d\rho + \int_{|\rho|>1} \operatorname{Re} p\Big(x, \frac{\rho\epsilon}{R}\Big) g(\rho)\, d\rho \ .$$

Set $\rho = [\rho] + \{\rho\}$ such that $[\rho] \in \mathbb{Z}$ and $\{\rho\} \in [0, 1)$. Using the sub-additivity of $\xi \mapsto \sqrt{\operatorname{Re} p(x, \xi)}$ we find

$$\operatorname{Re} p\Big(x, \frac{\rho\epsilon}{R}\Big) = \operatorname{Re} p\Big(x, \frac{([\rho] + \{\rho\})\epsilon}{R}\Big) \le 2\Big([\rho]^2 \operatorname{Re} p\Big(x, \frac{\epsilon}{R}\Big) + \operatorname{Re} p\Big(x, \frac{\{\rho\}\epsilon}{R}\Big)\Big) \ ,$$

and thus

$$\int_{-\infty}^{\infty} \operatorname{Re} p\Big(x, \frac{\rho\epsilon}{R}\Big) g(\rho)\, d\rho \le 2 \sup_{\|\epsilon\|\le 1} \operatorname{Re} p\Big(x, \frac{\epsilon}{R}\Big) \times \Big(\int_{-\infty}^{\infty} g(\rho) d\rho + \int_{-\infty}^{\infty} \rho^2 g(\rho) d\rho\Big) \ .$$

Passing to the supremum over $\|\epsilon\| \le 1$ on the left-hand side finishes the proof. □

The above Lemma and the definition of $H(x, R)$ (cf. Lemma 4.1) show that $H(x, R) \sim \sup_{\|\epsilon\|\le 1} \sup_{\|x-y\|\le 2R} |p(y, \epsilon/R)|$; the comparison constants do not depend on x.

Proposition 5.2 *Let $p(x, \xi)$ be as above, in particular, $\|p(\bullet, \xi)\|_\infty \le c(1 + \|\xi\|^2)$. Then the following formulae hold*

$$\beta_0 = \sup\Big\{\lambda \ge 0 : \lim_{\|\xi\|\to 0} \frac{\sup_{x\in\mathbb{R}^n} |p(x, \xi)|}{\|\xi\|^\lambda} = 0\Big\}$$

$$\underline{\beta}_0 \le \sup\Big\{\lambda \ge 0 : \liminf_{\|\xi\|\to 0} \frac{\sup_{x\in\mathbb{R}^n} |p(x, \xi)|}{\|\xi\|^\lambda} = 0\Big\}$$

$$\beta_\infty^x = \inf\Big\{\lambda > 0 : \lim_{\|\xi\|\to \infty} \frac{\sup_{\|x-y\|\le 2\|\xi\|^{-1}} |p(y, \xi)|}{\|\xi\|^\lambda} = 0\Big\}$$

$$\underline{\beta}_\infty^x \ge \inf\Big\{\lambda > 0 : \liminf_{\|\xi\|\to \infty} \frac{\sup_{\|x-y\|\le 2\|\xi\|^{-1}} |p(y, \xi)|}{\|\xi\|^\lambda} = 0\Big\}$$

Replacing $|p(\bullet,\bullet)|$ by $\mathrm{Re}\,p(\bullet,\bullet)$ *and* sup *by* inf *gives the analoguous formulae for the "*δ*"-indices of Definition* 4.5.

Proof. We will only prove the assertion for β^x_∞, the other proofs are similar or easier. From Lemma 5.1 it follows that

$$\beta^x_\infty = \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2\|\xi\|^{-1}, \|\epsilon\|\le 1} |p(y, \|\xi\|\epsilon)|}{\|\xi\|^\lambda} = 0\right\}$$
$$\ge \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\sup_{\|x-y\|\le 2\|\xi\|^{-1}} |p(y, \|\xi\|)|}{\|\xi\|^\lambda} = 0\right\} =: \alpha \ .$$

Now assume that $\lambda > \alpha$ and choose $\lambda > \lambda' > \alpha$. Then

$$\sup_{\|x-y\|\le 2\|\xi\|^{-1}} |p(y, \|\xi\|\epsilon)| \le c\|\epsilon\|^{\lambda'}\|\xi\|^{\lambda'} \le c\|\xi\|^{\lambda'} \quad \text{whenever} \quad \|\epsilon\|\cdot\|\xi\| \ge 1$$

Thus

$$\sup_{\|x-y\|\le 2\|\xi\|^{-1}} \ \sup_{\|\xi\|^{-1}\le\|\epsilon\|\le 1} \frac{|p(y, \|\xi\|\epsilon)|}{\|\xi\|^\lambda} \le c\|\xi\|^{\lambda'-\lambda} \to 0 \quad \text{as} \quad \|\xi\| \to \infty \ .$$

Since for $\|\xi\| \to \infty$

$$\sup_{\|x-y\|\le 2\|\xi\|^{-1}} \ \sup_{\|\epsilon\|\le\|\xi\|^{-1}} \frac{|p(y, \|\xi\|\epsilon)|}{\|\xi\|^\lambda} \le \sup_{\|x-y\|\le 2\|\xi\|^{-1}} \ \sup_{\|\eta\|\le 1} \frac{|p(y,\eta)|}{\|\xi\|^\lambda} \to 0 \ ,$$

we have $\lim_{\|\xi\|\to\infty} \sup_{\|x-y\|\le 2\|\xi\|^{-1}} \sup_{\|\epsilon\|\le 1} \frac{|p(y,\|\xi\|\epsilon)|}{\|\xi\|^\lambda} = 0$, therefore $\lambda \le \beta^x_\infty$ and–$\lambda > \alpha$ being arbitrary – $\alpha \le \beta^x_\infty$. □

Remark 5.3 In the above formulae for $\underline{\beta}_0$ and $\underline{\beta}^x_\infty$ a "=" sign cannot be expected. The supremum over all directions $\epsilon \in B_1(0)$ causes an averaging of the directional limiting behaviour. Clearly, the *limit* exists if and only if it exists for every coordinate direction $\|\xi\|^{-\lambda}|p(y, \|\xi\|e_j)|$ (this is easily seen by the subadditivity of $\xi \mapsto \sqrt{p(x,\xi)}$). This is clearly not the case for the limit inferior.

The results of the preceding Section 4 suggest also a nice stochastic interpretation: the limit inferior of the weighted vector-valued process can be different from the vector of the lim inf's of the weighted coefficient processes. Note, however, that this interpretation is based on the *assumption* that $p(x, \xi_j e_j)$ is *close* to the symbol of the jth coordinate process. This picture is certainly correct for Lévy processes, but it is still an open question in the general case.

In the absence of a quadratic form in the Lévy-Khinchine formula (4.1), i.e. if there is no continuous martingale part in the representation of the process $\{X_t\}_{t\geq 0}$, we can characterize the indices in terms of the Lévy kernel. Results of this type are known for Lévy processes and were, in fact, used to define the indices for appropriately normalized processes, see [4] for a thorough discussion. By normalization one understands removing a dominating drift and/or diffusion part of the process. While this can be done for Lévy processes without any harm – the drift being necessarily deterministic, the diffusion independent of the pure jump part – it is not possible for a general Feller process. *Thus we have to assume that, a priori, there is no diffusion part and that there is no dominating drift. We can express this by saying that* $Q(x) \equiv 0$ *and* $|\mathrm{Im}\, p(x,\xi)| \leq c_0 \mathrm{Re}\, p(x,\xi)$.

Proposition 5.4 *Assume that* $\mathbb{R}^n \ni \xi \mapsto p(x,\xi)$ *is for every* $x \in \mathbb{R}^n$ *continuous negative definite with Lévy characteristics* $(0, \ell(x), 0, N(x,dy))$, $\|p(\bullet,\xi)\|_\infty \leq c(1+\|\xi\|^2)$, *and such that* $|\mathrm{Im}\, p(x,\xi)| \leq c_0\, \mathrm{Re}\, p(x,\xi)$ *for some constant* c_0. *Then*

$$\beta_0 = \sup\left\{\lambda \geq 0 \ : \ \sup_{x\in\mathbb{R}^n} \int_{\|y\|>1} \|y\|^\lambda N(x,dy) < \infty\right\}$$

$$\bar{\delta}_0 = \sup\left\{\lambda \geq 0 : \inf_{x\in\mathbb{R}^n} \int_{\|y\|>1} \|y\|^\lambda N(x,dy) < \infty\right\}$$

$$\beta_\infty^x \leq \inf_{r>0} \inf\left\{\lambda > 0 : \sup_{\|x-z\|\leq r} \int_{\|y\|\leq 1} \|y\|^\lambda N(z,dy) < \infty\right\}$$

$$\beta_\infty^x \geq \inf\left\{\lambda > 0 : \int_{\|y\|\leq 1} \|y\|^\lambda N(x,dy) < \infty\right\}$$

$$\bar{\delta}_\infty^x \leq \inf\left\{\lambda > 0 : \int_{\|y\|\leq 1} \|y\|^\lambda N(x,dy) < \infty\right\}$$

$$\bar{\delta}_\infty^x \geq \sup_{r>0} \inf\left\{\lambda > 0 : \inf_{\|x-z\|\leq r} \int_{\|y\|\leq 1} \|y\|^\lambda N(z,dy) < \infty\right\}$$

Proof. Since the arguments for the above formulae are quite similar, we will only detail the proofs for the indices at the origin.

By our assumption on the symbol, $\mathrm{Re}\, p(x,\xi) \leq |p(x,\xi)| \leq (1+c_0)\ \mathrm{Re} p(x,\xi)$, and there is no need to distinguish between the modulus and the real part.

Assume that $\lambda < \beta_0$ and choose $\lambda < \lambda' < \beta_0$. By Proposition 5.2, $\mathrm{Re} p(x,\eta) \leq c\|\eta\|^{\lambda'}$ uniformly in x and (at least) for small values of $\|\eta\|$. Denote by $k_\lambda(\eta)$ the density satisfying

$$\int_{\mathbb{R}^n} (1-\cos y\cdot\eta) k_\lambda(\eta)\, d\eta = \|y\|^\lambda, \quad y \in \mathbb{R}^n\ ,$$

and observe that $k_\lambda(\eta) \sim \|\eta\|^{-\lambda-n}$, see [3, III.18.23]. Then

$$\begin{aligned}\int_{\|y\|\geq 1} \|y\|^\lambda N(x,dy) &= \int_{\|y\|\geq 1}\int_{\mathbb{R}^n} (1-\cos y\cdot\eta) k_\lambda(\eta)\, d\eta N(x,dy)\\ &\leq c_{\lambda,n}\int_{\mathbb{R}^n}\big(\operatorname{Re} p(x,\eta)\wedge N(x,B_1^c(0))\big)\|\eta\|^{-\lambda-n}\,d\eta\\ &\leq c'_{\lambda,n}\int_{\|\eta\|\leq 1}\|\eta\|^{(\lambda'-\lambda)-n}\,d\eta\\ &\quad + c'_{\lambda,n}\int_{\|\eta\|>1}\|\eta\|^{-\lambda-n} d\eta N(x,B_1^c(0))\end{aligned}$$

uniformly for all x, since by Lemma 2.1

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n} N(x,B_1^c(0))\\ &\quad\leq 2\sup_{x\in\mathbb{R}^n}\int_{\|y\|\geq 1}\frac{\|y\|^2}{1+\|y\|^2}N(x,dy)\leq 2c\int_{\mathbb{R}^n}(1+\|\xi\|^2)g_n(\xi)\,d\xi<\infty\ .\end{aligned}$$

This shows that $\lambda\leq\sup\{\lambda>0:\sup_{x\in\mathbb{R}^n}\int_{\|y\|>1}\|y\|^\lambda N(x,dy)<\infty\}$, and since $\lambda<\beta_0$ was arbitrary, also $\beta_0\leq\sup\{\lambda>0:\sup_{x\in\mathbb{R}^n} \times\int_{\|y\|>1}\|y\|^\lambda N(x,dy)<\infty\}$.

Replacing in the above calculations β_0 by $\bar\delta_0$ and $\operatorname{Re} p(x,\xi)$ by $\inf_{x\in\mathbb{R}^n}\operatorname{Re} p(x,\xi)$ proves $\bar\delta_0\leq\inf\{\lambda>0:\inf_{x\in\mathbb{R}^n}\int_{\|y\|>1}\|y^\lambda\| N(x,dy)<\infty\}$.

Conversely, assume that $\lambda\leq 2$ is such that

$$\inf_{x\in\mathbb{R}^n}\int_{\|y\|>1}\|y\|^\lambda N(x,dy)\leq\sup_{x\in\mathbb{R}^n}\int_{\|y\|>1}\|y\|^\lambda N(x,dy)<\infty\ .$$

Then, since we are only interested in small values of $\|\xi\|$, say $\|\xi\|\leq 1$,

$$\begin{aligned}\operatorname{Re} p(x,\xi) &= \int_{y\neq 0}(1-\cos y\cdot\xi)N(x,dy)\\ &\leq\int_{0<\|y\|\leq 1}\|y\|^2 N(x,dy)\|\xi\|^2+\int_{\|y\|>1}\|y\|^\lambda N(x,dy)\|\xi\|^\lambda\\ &\leq\left(\sum_{j=1}^n\int_{\|y\|\leq 1}\frac{1-\cos y\cdot e_j}{1-\cos 1}N(x,dy)\|\xi\|^{2-\lambda}\right.\\ &\qquad\left.+\int_{\|y\|>1}\|y\|^\lambda N(x,dy)\right)\|\xi\|^\lambda\\ &\leq c\left(\sum_{j=1}^n\operatorname{Re} p(x,e_j)\|\xi\|^{2-\lambda}+\int_{\|y>1\|}\|y\|^\lambda N(x,dy)\right)\|\xi\|^\lambda\end{aligned}$$

uniformly for all x. By Proposition 5.2, $\lambda\leq\bar\delta_0$ and $\lambda\leq\beta_0$.

The inequalities for β_∞^x and $\overline{\delta}_\infty^x$ are easily derived from the inequalities

$$\int_{\|y\|\le 1} \|y\|^\lambda N(x,dy) \le c_\lambda \int_{\|\eta\|\ge 1} \operatorname{Re} p(x,\eta) \|\eta\|^{-\lambda-n}\, d\eta + c'$$

and

$$\operatorname{Re} p(x,\xi) \le c_\lambda \int_{\|y\|\le 1} \|y\|^\lambda N(x,dy) \|\xi\|^\lambda + c'$$

which are proved as those used above. Note that c_λ and c' are independent of x. □

Let us conclude this section with some examples.

Example 5.5 (1) Lévy processes $p(x,\xi) = \psi(\xi)$. In order to evade all normalization problems we assume $|\operatorname{Im} \psi(\xi)| \le c_0 \operatorname{Re} \psi(\xi)$, but we do allow a Gaussian part. In this case we have

$$\beta_0 = \overline{\delta}_0, \quad \underline{\beta}_0 = \delta_0, \quad \beta_\infty = \beta_\infty^x = \overline{\delta}_\infty^x, \quad \text{and} \quad \delta_\infty = \delta_\infty^x = \underline{\beta}_\infty^x \ .$$

Because of Proposition 5.2 β_∞ is the Blumenthal-Getoor index, and β_∞ and β_0 can be calculated via

$$\beta_\infty = \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\} \ ,$$

$$\beta_0 = \sup\left\{\lambda \ge 0 : \lim_{\|\xi\|\to 0} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\} \ .$$

and

$$\delta_\infty = \inf\left\{\lambda > 0 : \liminf_{\|\xi\|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\} \ ,$$

$$\delta_0 = \sup\left\{\lambda \ge 0 : \liminf_{\|\xi\|\to 0} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\} \ .$$

In particular, Theorems 4.3 and 4.6 become trichotomies and distinguish *properly* between the various modes of limiting behaviour: convergence, oscillation, and divergence. This is just the case considered by Pruitt [27].

Note, that for symmetric α-stable processes all of the above indices coincide and equal α.

(2) Stable-like processes: long-time behaviour. A Feller process whose generator has a symbol of the form $p(x,\xi) = \|\xi\|^{2a(x)}$, $x, \xi \in \mathbb{R}^n$ is called *stable-like*. This notion seems to be due to Bass [1],

constructions of such processes were given by Bass [1], Tsuchyia [35], Negoro [25], and Kikuchi &Negoro [22]. We will assume that the exponent function is uniformly bounded, i.e. that $0 < a_0 \le a(x) \le a_\infty < 1$ – this is a quite common assumption in the existing literature.

Clearly, we have

$$\beta_0 = \underline{\beta}_0 = a_0 \quad \text{and} \quad \bar{\delta}_0 = \delta_0 = a_\infty \ .$$

Replacing the modulus $\|\xi\|$ by an arbitrary real-valued continuous negative definite functions ψ, $\psi(0) = 0$, leads to symbols of the type $p(x,\xi) = (1+\psi(\xi))^{a(x)} - 1 \sim \psi(\xi)^{a(x)}$. Such symbols were considered e.g. by [22]. In this case one has

$$\beta_0 = \beta_0(\psi) a_0, \quad \bar{\delta}_0 = \bar{\delta}_0(\psi) a_\infty \ ,$$

where $\beta_0(\psi), \bar{\delta}_0(\psi)$ are the indices for ψ. Similar statements hold for $\underline{\beta}_0, \delta_0$.

(3) Stable-like processes: short-time behaviour. Let us assume that $p(x,\xi) = \psi(\xi)^{a(x)}$ (or, alternatively, $= (1+\psi(\xi))^{a(x)} - 1$) where $\psi : \mathbb{R}^n \to \mathbb{C}$, $\psi(0) = 0$, is continuous negative definite such that $|\mathrm{Im}\,\psi(\xi)| \le c_0 \mathrm{Re}\,\psi(\xi)$. The exponent function $a : \mathbb{R}^n \to (0,1)$ is assumed to be uniformly bounded, $0 < a_0 \le a(x) \le a_\infty < 1$.

Note that under these assumptions

$$\begin{aligned} |\mathrm{Im}\, p(x,\xi)| = |\mathrm{Im}\,\psi(\xi)^{a(x)}| &= |\psi(\xi)|^{a(x)} |\sin\big(a(x)\arg\psi(\xi)\big)| \\ &\le c|\psi(\xi)|^{a(x)} \cos\big(a(x)\arg\psi(\xi)\big) = c\mathrm{Re}\,\psi(\xi)^{a(x)} \\ &= c\mathrm{Re}\, p(x,\xi) \end{aligned}$$

because we know $|\sin\big(a(x)\arg\psi(\xi)\big)| \le |\sin \ \big(a_\infty \arg\psi(\xi)\big)| \le c\cos\big(a_\infty \arg\psi(\xi)\big)) \le c\cos\big(a(x)\arg\psi(\xi)\big))$ with a constant c depending only on the bounds of the exponent function $a(\bullet)$ and the constant c_0.

If, in addition, $a(\bullet)$ *has locally a modulus of continuity of order* $\simeq o(|1/\log h|)$, $(h \to 0)$, *we have*

$$\beta_\infty^x = \bar{\delta}_\infty^x = \beta_\infty(\psi) \cdot a(x) \quad \textit{and} \quad \underline{\beta}_\infty^x = \delta_\infty^x = \delta_\infty(\psi) \cdot a(x) \tag{5.2}$$

where $\beta_\infty(\psi), \delta_\infty(\psi)$ *stand for the indices of* $\psi(\xi)$, *see Example* (1).

Proof. Because of the particular modulus of continuity we find for every x some $h = h(x)$ such that

$$|a(x) - a(y)| \le \phi_x(\|x-y\|) \quad \text{for all} \quad y \in B_h(x)$$

for some monotonically increasing function $\phi_x(\bullet)$ satisfying $\lim_{h\to 0} \phi_x(h) \log\frac{1}{h} = 0$ for every $x \in \mathbb{R}^n$. For all $\xi, x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\left|\psi(\xi)^{a(x)} - \psi(\xi)^{a(y)}\right| &= \left|\psi(\xi)^{a(x)}\right| \cdot \left|1 - \psi(\xi)^{a(y)-a(x)}\right| \\ &\le \left|\psi(\xi)^{a(x)}\right| \left(1 + |1+\psi(\xi)|^{|a(y)-a(x)|}\right) \\ &\le \left|\psi(\xi)^{a(x)}\right| \left(1 + |1+\psi(\xi)|^{\phi_x(|x-y|)}\right) \ .\end{aligned}$$

If $\|x-y\| \le c'R$ and $\xi = \epsilon/R$, $\|\epsilon\| \le 1$, we get

$$\begin{aligned}&\left|1+\psi\left(\frac{\epsilon}{R}\right)\right|^{\phi_x(\|x-y\|)} \le \exp\left(\phi_x(c'R)\log\left(c_\psi(1+R^{-2})\right)\right) \\ &\le \exp\left(2\phi_x(c'R)\log\left(\frac{c'\sqrt{c_\psi}}{c'R}\right)\right) \\ &= \exp\left(2\phi_x(c'R)\Big[\log(c'\sqrt{c_\psi}) + \log\left(1/(c'R)\right)\Big]\right) \\ &\to 1\end{aligned}$$

as $R \to 0$. Thus,

$$\lim_{R\to 0}\sup_{\|x-y\|\le c'R} R^\lambda \left|\psi\left(\frac{\epsilon}{R}\right)^{a(x)}\right| = \lim_{R\to 0} R^\lambda \left|\psi\left(\frac{\epsilon}{R}\right)^{a(x)}\right| = 0, \quad \text{resp.,} \quad \infty$$

according to $\lambda > \beta_\infty(\psi)\cdot a(x)$, resp., $\lambda < \beta_\infty(\psi)\cdot a(x)$. This shows $\beta_\infty^x = \beta_\infty a(x) = \bar{\delta}_\infty^x$.

Lemma 5.1 (applied to $\psi(\xi)$ rather than $p(x,\xi)$) and $|\mathrm{Im}\,\psi(\xi)| \le c_0\ \mathrm{Re}\,\psi(\xi)$ give

$$\begin{aligned}\sup_{\|\epsilon\|\le 1}\mathrm{Re}\,\psi\left(\frac{\epsilon}{R}\right) &\sim \sup_{\|\epsilon\|\le 1}\left|\psi\left(\frac{\epsilon}{R}\right)\right| \\ &\sim \sup_{\|\epsilon\|\le 1}\left[\int_{-\infty}^{\infty}\psi\left(\frac{\rho\epsilon}{R}\right)g(\rho)d\rho + \mathrm{Re}\,\psi\left(\frac{\epsilon}{R}\right)\right]\end{aligned}$$

hence $\delta_\infty^x = \underline{\beta}_\infty^x$ and it remains to check that $\delta_\infty^x = \delta_\infty(\psi)\cdot a(x)$. This, however, follows immediately from

$$\begin{aligned}\liminf_{R\to 0}\left(R^\lambda \sup_{\|\epsilon\|\le 1}\left|\psi\left(\frac{\epsilon}{R}\right)^{a(x)}\right|\right) &\sim \liminf_{R\to 0}\left(R^\lambda\sup_{\|\epsilon\|\le 1}\left(\mathrm{Re}\,\psi\left(\frac{\epsilon}{R}\right)\right)^{a(x)}\right) \\ &= \left(\liminf_{R\to 0}\left(R^{\lambda/a(x)}\sup_{\|\epsilon\|\le 1}\mathrm{Re}\,\psi\left(\frac{\epsilon}{R}\right)\right)\right)^{a(x)} \ ,\end{aligned}$$

(note that $\arg\psi(\bullet) \in [\delta - \frac{\pi}{2}, \frac{\pi}{2} - \delta]$ for some $\delta = \delta(c_0) > 0$), and we are done. □

If we choose, in particular, $\psi(\xi) = \|\xi\|^2$ we get $\beta_\infty(\psi) = \delta_\infty(\psi) = 2$ and the above calculations show that

$$\beta_\infty^x = \bar{\delta}_\infty^x = \underline{\beta}_\infty^x = \delta_\infty^x = 2a(x) \ ,$$

and Theorem 4.6 specializes to the short-time asymptotics for stable-like processes treated by Negoro [25, Theorem 2.1]. Note, however, that our assumptions on the exponent $a(x)$ are less restrictive than in [25].

(4) General Feller processes. Assume that the symbol $-p(x,\xi)$ is given by (4.1) and that it satisfies $\|p(\bullet,\xi)\|_\infty \le c(1+\|\xi\|^2)$ and $|\operatorname{Im} p(x,\xi)| \le c_0 \operatorname{Re} p(x,\xi)$. By $U = U(x)$ we denote some neighbourhood of $x \in \mathbb{R}^n$ that may depend on the particular choice of x. If either

$$(a)\begin{cases} \text{for all } x \text{ there exist continuous negative definite functions } \psi_U \\ \text{such that } \sup_{x\in U}\big(1+|p(x,\xi)|\big) \sim 1+\psi_U(\xi), \quad \xi\in\mathbb{R}^n \end{cases}$$

or

$$(b)\begin{cases} x\mapsto p(x,\xi) \text{ is Lipschitz continuous such that locally} \\ |p(x,\xi)-p(y,\xi)| \le c_U(1+\psi_U(\xi))\|x-y\|, \quad y\in U \\ \text{holds for (locally) fixed continuous negative definite} \\ \text{functions } \psi_U \text{ satisfying } \beta_\infty(\psi_U)\le 1 \end{cases}$$

or

$$(c)\begin{cases} x\mapsto p(x,\xi) \text{ is Lipschitz continuous such that locally} \\ |p(x,\xi)-p(y,\xi)| \le c_U(1+\psi_U(\xi))\|x-y\|, \quad y\in U \\ \text{holds for (locally) fixed continuous negative definite functions } \psi_U \\ \text{such that } \quad c\,\dfrac{\psi_U(\xi)}{\|\xi\|} \le 1+p(x,\xi) \quad \text{for large } \|\xi\| \end{cases}$$

or

$$(d)\begin{cases} 1+p(x,\xi)\sim(1+\|\xi\|^2)^{a(x)} \text{ with an exponent function } a(x) \text{ which is} \\ \text{bounded } 0<a_0\le a(x)\le a_\infty<1 \text{ and has locally some modulus} \\ \text{of continuity of order } \simeq o(|1/\log h|) \text{ as } h\to 0 \end{cases}$$

is satisfied, then

$$\beta_\infty^x = \overline{\delta}_\infty^x = \inf\left\{\lambda>0 : \lim_{\|\xi\|\to\infty} \frac{\operatorname{Re} p(x,\xi)}{\|\xi\|^\lambda} = 0\right\},$$

$$\delta_\infty^x = \underline{\beta}_\infty^x = \inf\left\{\lambda>0 : \liminf_{\|\xi\|\to\infty} \sup_{\|\epsilon\|\le 1} \frac{\operatorname{Re} p(x,\|\xi\|\epsilon)}{\|\xi\|^\lambda} = 0\right\}.$$

Note that (d) encompasses the results by Kikuchi & Negoro [22, Theorem 3.9].

Proof. The sufficiency of condition (a) is trivial, (b) and (c) follow easily from Proposition 5.2 and the observation that for $\|x-y\| \le R$ and $\epsilon \in \mathbb{R}^n$, $\|\epsilon\| \le 1$

$$\begin{aligned} R^\lambda \left| p\left(y, \frac{\epsilon}{R}\right)\right| \le & R^\lambda \left| p\left(x, \frac{\epsilon}{R}\right)\right| + R^\lambda \left| p\left(y, \frac{\epsilon}{R}\right) - p\left(x, \frac{\epsilon}{R}\right)\right| \\ \le & R^\lambda \left| p\left(x, \frac{\epsilon}{R}\right)\right| + c_U R^{\lambda+1}\left(1+\psi_U\left(\frac{\epsilon}{R}\right)\right) . \end{aligned}$$

The conditions (b), (c) ensure that the second member on the right is finite as $R \to 0$.

For (d) we use

$$R^\lambda \left| p\left(y, \frac{\epsilon}{R}\right) - p\left(x, \frac{\epsilon}{R}\right)\right| = R^\lambda \left| p\left(x, \frac{\epsilon}{R}\right)\right| \left(1 - \left|\frac{p\left(y, \frac{\epsilon}{R}\right)}{p\left(x, \frac{\epsilon}{R}\right)}\right|\right)$$

and, by the continuity property of the exponent, (see Example (3) for details)

$$\left|\frac{p\left(y, \frac{\epsilon}{R}\right)}{p\left(x, \frac{\epsilon}{R}\right)}\right| \le c\left(1 + \left\|\frac{\epsilon}{R}\right\|^2\right)^{|a(y)-a(x)|} \to 1 \quad \text{as} \quad \mathrm{R} \to 0 . \qquad \square$$

6 Proof of Lemma 4.1

We will now present the proof of Lemma 4.1. Since the proof is rather lengthy we split it up into several steps.

Throughout this section, $R > 0$ will be a fixed constant and we will use the following notations without further notice: $\chi = \chi_R$ is a cut-off function satisfying

$$\chi \in C_c^\infty(\mathbb{R}^n), \quad \mathbf{1}_{B_R(0)} \le \chi \le \mathbf{1}_{B_{2R}(0)}, \quad \chi(y) = \chi(-y) . \tag{6.1}$$

Let $\Delta X_t := X_t - X_{t-}$ denote the jump at time t. We need the following stopping time for $S > 2R > 0$

$$\sigma := \sigma_S^x := \inf\left\{t > 0 : \|X_t - x\| > S\right\} , \tag{6.2}$$

the usual notation for the maximal process

$$(X_\bullet - x)_t^* := \sup_{s \le t} \|X_s - x\| \tag{6.3}$$

and similar *-expressions which should be self-explanatory.

Let us start with the proof of (4.3).

Lemma 6.1 *Let* $(A, D(A))$ *be a Feller generator such that* $C_c^\infty(\mathbb{R}^n) \subset D(A)$ *and* $A|C_c^\infty(\mathbb{R}^n) = -p(x, D)$. *Assume that the symbol is given by* (4.1), *has Lévy characteristics* $(0, \ell(x), Q(x), N(x, dy))$, *and satisfies* $\|p(\bullet, \xi)\|_\infty \le c(1 + \|\xi\|^2)$. *Then for all* $S > 2R > 0$ *and with* σ *of* (6.2)

$$\mathbb{P}^x\big((X^\sigma_\bullet - x)^*_t \ge 2R\big)$$
$$\le cnt \sup_{\|y-x\|\le S} \sup_{\|\epsilon\|\le 1} \left(\int_{-\infty}^{\infty} \operatorname{Re} p\Big(y, \frac{\rho\epsilon}{2R}\Big) g(\rho) d\rho + \left| p\Big(y, \frac{\epsilon}{2R}\Big) \right| \right)$$

with an absolute constant $c > 0$ *and the function* $g(\rho) = g_1(\rho)$ *from* (2.5), (2.6).

Proof. Since $\{X_t\}_{t\ge 0}$ is a homogeneous diffusion with jumps (see Theorem 3), its jump measure $\mu^X(\bullet, ds, dy)$ has $N(X_s, -dy)ds$ as compensator. The semimartingale property is preserved under stopping, hence $X^\sigma_\bullet$ is again a semimartingale, and its characteristics (w.r.t. the cut-off function $\chi = \chi_R$) are given by

$$B^\sigma_\bullet = \int_0^{\bullet\wedge\sigma} \left[\ell(X_s) + \int_{y\ne 0} \left(\frac{y}{1 + \|y\|^2} - \chi(y)y \right) N(X_s, dy) \right] ds;$$

$$C^\sigma_\bullet = 2\int_0^{\bullet\wedge\sigma} Q(X_s)ds; \quad \mathbf{1}_{[0,\sigma]}(s)N(X_s, -dy)ds$$

(to be read componentwise). This can be easily checked using the criterion in [21, Theorem II.2.21].

Observe that for the stopping time $\tau_R := \inf\{t > 0 : \|\Delta X^\sigma_t\| > R\}$ of the stopped process $X^\sigma_\bullet$

$$\mathbb{P}^x\big((X^\sigma_\bullet - x)^*_t \ge 2R\big) \le \mathbb{P}^x\big((X^\sigma_\bullet - x)^*_t \ge 2R, \tau_R > t\big) + \mathbb{P}^x\big(\tau_R \le t\big) \tag{6.4}$$

Beginning with the second member, we will estimate the terms of the right-hand side separately. Choose some strictly increasing function $\imath : \mathbb{R} \to [1, 2]$ and set

$$I^\imath_t := \int_0^{t\wedge\sigma} \int_{\|y\|\ge R} \imath(\|y\|)\mu^X(\bullet, ds, dy) = \sum_{s, \Delta X^\sigma_s \ge R} \imath(\|\Delta X^\sigma_s\|) \ .$$

Note that the integral is well-defined, since a.s. only finitely many jumps of a fixed size occur in finite time. On the set $\{\tau_R \le t\}$ at least one jump of size $\ge R$ occurs during $[0, \tau_R) \cap [0, t]$, thus $I^\imath_t \ge \imath(R)$. This shows

$$
\begin{aligned}
\mathbb{P}^x\big(\tau_R \le t\big) &\le \mathbb{P}^x\big(I_t^\imath \ge \imath(R)\big) \le \frac{1}{\imath(R)}\,\mathbb{E}^x\left(\int_0^{t\wedge\sigma}\int_{\|y\|\ge R} \imath(\|y\|)\mu^X(\bullet, ds, dy)\right)\\
&= \frac{1}{\imath(R)}\,\mathbb{E}^x\left(\int_0^{t\wedge\sigma}\int_{\|y\|\ge R} \imath(\|y\|)N(X_s, dy)\ ds\right)\\
&\le\ 2t\mathbb{E}^x\left(\sup_{s<t\wedge\sigma} N(X_s, B_R^c(0))\right)\\
&\le 2t \sup_{\|y-x\|\le S} N(y, B_R^c(0))\\
&\le\ c_n t \sup_{\|y-x\|\le S}\left(\sum_{j=1}^n \int_{z\ne 0} \frac{\left(\frac{z_j}{2R}\right)^2}{1+\left(\frac{z_j}{2R}\right)^2} N(y, dz)\right)\\
&\le\ c_n t \sup_{\|y-x\|\le S}\ \sup_{\|\epsilon\|\le 1} \int_{-\infty}^{\infty} \operatorname{Re} p\Big(y, \frac{\rho\epsilon}{2R}\Big) g(\rho)\, d\rho\ , \qquad (6.5)
\end{aligned}
$$

where we just used the properties of the compensator of a random measure and applied the technique of Lemma 2.1, (2.5), (2.6) with $g(\rho) = g_1(\rho)$.

Let us now turn to the first summand in (6.4). Recall that

$$
\check{X}_t := X_t - J_t = X_t - \sum_{s\le t}(1-\chi(\Delta X_s))\Delta X_s
$$

from (3.3) is a semimartingale with bounded jumps. Thus $\check{X}^\sigma_\bullet$ has the semimartingale characteristics

$$
\begin{aligned}
B^\sigma_\bullet &= \int_0^{\bullet\wedge\sigma}\left[\ell(X_s) + \int_{y\ne 0}\left(\frac{y}{1+\|y\|^2} - \chi(y)y\right) N(X_s, dy)\right] ds\\
C^\sigma_\bullet &= 2\int_0^{\bullet\wedge\sigma} Q(X_s)\, ds; \quad\ \chi(y)\mathbf{1}_{[0,\sigma]}(s)N(X_s, -dy)\, ds
\end{aligned}
$$

(to be read componentwise). Therefore, for all $u = (u_1, \ldots, u_n)$, where $u_j \in C_b^2(\mathbb{R})$ depends only on x_j (we will write, however, $u_j(x)$, $u_j'(x)$ rather than the clumsier $u_j(x_j)$, $\frac{d}{dx_j}u_j(x_j)$,...) the process

$$
\check{M}_t := u(\check{X}^\sigma_t - x) - \int_0^{t\wedge\sigma} N_s\, ds
$$

where

$$
\begin{aligned}
N_s^{(j)} := \Big[\ell^{(j)}(X_s) &+ \int_{y\ne 0}\left(\frac{y_j}{1+\|y\|^2} - \chi(y)y_j\right) N(X_s, dy)\Big]\\
&\times u_j'(\check{X}_s - x) + q_{jj}(X_s)u_j''(\check{X}_s - x)\\
&\quad + \int_{y\ne 0}(u_j(\check{X}_s - x - y) - u_j(\check{X}_s - x) + \chi(y)y_j u_j'(\check{X}_s - x))\\
&\quad \times \chi(y)N(X_s, dy)
\end{aligned}
$$

is a local martingale. Since $u_j \in C_b^2(\mathbb{R})$, one can estimate $|N_s^{(j)}|$ by expressions of the form *const.*$(\|u_j\|_\infty + \|u_j'\|_\infty + \|u_j''\|_\infty)$, see e.g. the calculations leading to (2.9). Thus, $\check{M}_t$ is an L^2_{loc}-*martingale*, see [26, Theorem I.47]. Let us now fix a $u = (u_1, \ldots, u_n)$ that satisfies also

$$u_j \in C_b^2(\mathbb{R}^n), \quad u_j(0) = 0, \quad |u_j(x) - u_j(y)| \leq |x_j - y_j|, \quad 1 \leq j \leq n \ . \tag{6.6}$$

We will have to distinguish two more cases. Set

$$B := \left\{ \omega \in \Omega : \int_0^{t \wedge \sigma(\omega)} \|N_s(\omega)\| ds \ \leq \ R \right\} \ .$$

Then, by the triangle inequality, and the fact that $\check{X}^\sigma_\bullet = X^\sigma_\bullet$ on $\{\tau_R > t\}$,

$$\begin{aligned}
&\mathbb{P}^x\big(\big(u(X^\sigma_\bullet - x)\big)^*_t \geq 2R, \tau_R > t; B\big) \\
&\qquad \leq \mathbb{P}^x\left(\left(u(\check{X}^\sigma_\bullet - x) - \int_0^{\bullet\wedge\sigma} N_s ds\right)^*_t \geq R, \tau_R > t; B\right) \\
&\qquad \leq \mathbb{P}^x(\check{M}^*_{t\wedge\sigma} \geq R) \ \leq \ \frac{1}{R^2}\mathbb{E}^x\big(\|\check{M}^\sigma_t\|^2\big)
\end{aligned}$$

where we applied Kolmogorov's maximal inequality to the martingale $\check{M}^\sigma_t = u(\check{X}^\sigma_t - x) - \int_0^{t\wedge\sigma} N_s\, ds$.

Since $t \mapsto \int_0^{t\wedge\sigma} N_s ds$ is a continuous process with paths of finite variation in finite time, we get for the quadratic variation (cf. [26, Section II.6, Theorems II.22, 25]) (to be read componentwise!)

$$\begin{aligned}
[\check{M}_\bullet, \check{M}_\bullet]^\sigma &= \left[u(\check{X}_\bullet - x) - \int_0^\bullet N_s\, ds, u(\check{X}_\bullet - x) - \int_0^\bullet N_s\, ds\right]^\sigma \\
&= [u(\check{X}_\bullet - x), u(\check{X}_\bullet - x)]^\sigma \\
&\quad + 2\left[u(\check{X}_\bullet - x), \int_0^\bullet N_s\, ds\right]^\sigma + \left[\int_0^\bullet N_s\, ds, \int_0^\bullet N_s\, ds\right]^\sigma \\
&= [u(\check{X}_\bullet - x), u(\check{X}_\bullet - x)]^\sigma \ .
\end{aligned}$$

The functions u_j are Lipschitz, hence we get from the very definition of the quadratic variation of a semimartingale, cf. [26, Theorem II.22]

$$[\check{M}^{(j)}_\bullet, \check{M}^{(j)}_\bullet]^\sigma_t = [u_j(\check{X}_\bullet - x), u_j(\check{X}_\bullet - x)]^\sigma_t \leq [\check{X}^{(j)}_\bullet, \check{X}^{(j)}_\bullet]^\sigma_t \ .$$

Taking expectations and observing that $\mathbb{E}^x\left(\|\check{M}^\sigma_t\|^2\right) = \mathbb{E}^x(\sum_{j=1}^n \big[\check{M}^{(j)}_\bullet, \check{M}^{(j)}_\bullet\big]^\sigma_t)$, e.g. [26, p. 66 Corollary 3], yields

$$\mathbb{P}^x\Big(\big(u(X^\sigma_\bullet - x)\big)^*_t \geq 2R, \tau_R > t; B\Big) \leq \frac{1}{R^2}\sum_{j=1}^n \mathbb{E}^x([\check{X}^{(j)}_\bullet, \check{X}^{(j)}_\bullet]^\sigma_t) \ .$$

Recall that the canonical representation of the semimartingale $\check{X}^\sigma_\bullet$, cf. (3.6), is

$$\check{X}^\sigma_t = X_0 + B^\sigma_t + X^{c,\sigma}_t + \int_0^{t\wedge\sigma}\int_{\mathbb{R}^n} \chi(y)y \ \widetilde{\mu}(\bullet, ds, dy)$$

where $B_\bullet$ is a continuous finite-variation process (since $X_\bullet$ is a diffusion with jumps), $X^c_\bullet$ is the continuous martingale part, and $\widetilde{\mu}(\bullet, ds, dy) := \mu^X(\bullet, ds, dy) - N(X_s, -dy)\, ds$ is the compensated jump measure. By the argument already used above for the quadratic variations we find (to be read componentwise)

$$\big[\check{X}_\bullet, \check{X}_\bullet\big]^\sigma = \langle X^c_\bullet, X^c_\bullet\rangle^\sigma + \left[\int_0^\bullet\int_{\mathbb{R}^n} \chi(y)y\widetilde{\mu}(\bullet, ds, dy), \int_0^\bullet\int_{\mathbb{R}^n} \chi(y)y\widetilde{\mu}(\bullet, ds, dy)\right]^\sigma \ .$$

Now observe that for every $j = 1, \ldots, n$

$$\mathbb{E}^x\left(\left[\int_0^\bullet\int_{\mathbb{R}^n} y_j\chi(y)\widetilde{\mu}(\bullet, ds, dy), \int_0^\bullet\int_{\mathbb{R}^n} y_j\chi(y)\widetilde{\mu}(\bullet, ds, dy)\right]^\sigma_t\right) = \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\int_{\mathbb{R}^n} y_j^2\chi^2(y)N(X_s, dy)ds\right) \ ,$$

see, [12, p. 62 (3.9)] or [21, Theorem II.1.33], and that by the definition,

$$\langle X^{(j),c}_\bullet, X^{(j),c}_\bullet\rangle = 2\int_0^\bullet q_{jj}(X_s)\, ds \ .$$

Combining these remarks, we have found

$$\mathbb{P}^x(\big(u(X^\sigma_\bullet - x)\big)^*_t \geq 2R, \tau_R > t; B)$$

$$\leq \frac{2}{R^2}\sum_{j=1}^n \mathbb{E}^x\left(\int_0^{t\wedge\sigma} q_{jj}(X_s)\, ds\right) + \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\int_{y\neq 0} \frac{\|y\|^2}{R^2}\chi^2(y)N(X_s, dy)\, ds\right)$$

$$\leq 8\sum_{j=1}^n \mathbb{E}^x\left[\int_0^{t\wedge\sigma}\left(\frac{e_j}{2R}\cdot Q(X_s)\frac{e_j}{2R} + \int_{y\neq 0}\left(y\cdot\frac{e_j}{2R}\right)^2\chi^2(y)N(X_s, dy)\right)ds\right].$$

$$\leq 8\sum_{j=1}^n \mathbb{E}^x\left[\int_0^{t\wedge\sigma}\left(\frac{e_j}{2R}\cdot Q(X_s)\frac{e_j}{2R} + \int_{y\neq 0}\frac{1-\cos\frac{y\cdot e_j}{2R}}{1-\cos 1}\chi^2(y)N(X_s, dy)\right)ds\right] \ .$$

where the elementary inequality $\alpha^2(1-\cos 1) \le 1-\cos\alpha$ for $|\alpha| \le 1$ and $\operatorname{supp}\chi \subset B_{2R}(0)$ entered in the last line. Recalling the Lévy-Khinchine formula 4.1 for $p(x,\xi)$ we find

$$\begin{aligned} &\mathbb{P}^x\Big(\big(u\big(X_\bullet^\sigma - x\big)\big)_t^* \ge 2R, \tau_R > t; B\Big) \\ &\le ct\sum_{j=1}^n \mathbb{E}^x\Big(\sup_{s<t\wedge\sigma} \operatorname{Re} p\Big(X_s, \frac{e_j}{2R}\Big)\Big) \\ &\le cnt \sup_{\substack{\|y-x\|\le S\\ \|\epsilon\|\le 1}} \operatorname{Re} p\Big(y, \frac{\epsilon}{2R}\Big) \ . \end{aligned} \tag{6.7}$$

We still have to handle the case when $\omega \in B^c$. By the Markov inequality we find

$$\begin{aligned} \mathbb{P}^x\Big(\big(u(X_\bullet^\sigma - x)\big)_t^* \ge 2R, \tau_R > t; B^c\Big) &\le \mathbb{P}^x(B^c) = \mathbb{P}^x\Big(\int_0^{t\wedge\sigma} \|N_s\|\, ds > R\Big) \\ &\le \frac{1}{R}\sum_{j=1}^n \mathbb{E}^x\Big(\int_0^{t\wedge\sigma} |N_s^{(j)}|\, ds\Big) \ . \end{aligned}$$

Adding this and (6.7) gives

$$\begin{aligned} &\mathbb{P}^x((u(X_\bullet^\sigma - x))_t^* \ge 2R, \tau_R > t) \\ &\quad \le cnt \sup_{\substack{\|y-x\|\le S\\ \|\epsilon\|\le 1}} \operatorname{Re} p\Big(y, \frac{\epsilon}{2R}\Big) + \frac{1}{R}\sum_{j=1}^n \mathbb{E}^x\Big(\int_0^{t\wedge\sigma} |N_s^{(j)}|\, ds\Big) \end{aligned} \tag{6.8}$$

Since the process N_s is norm-bounded by $const. \sum_{|\alpha|=1,2} \|\partial^\alpha u\|_\infty$, (recall that the cut-off function χ appearing in the definition of N_s has compact support!) we can choose a sequence u^k of functions satisfying (6.6) and such that $|u_j^k(x) - x_j|$ decreases to 0 as $k \to \infty$ while the first and second-order derivatives stay bounded. Using dominated convergence on the right-hand side of (6.8) and monotone convergence on the left, gives

$$\begin{aligned} &\mathbb{P}^x((X_\bullet^\sigma - x)_t^* \ge 2R, \tau_R > t) \le cnt \sup_{\substack{\|y-x\|\le S\\ \|\epsilon\|\le 1}} \operatorname{Re} p\Big(y, \frac{\epsilon}{2R}\Big) \\ &+ \frac{2}{2R}\sum_{j=1}^n \mathbb{E}^x \times \Bigg(\int_0^{t\wedge\sigma} \Bigg|\Bigg[\ell(X_s) + \int_{y\ne 0}\Big(\frac{y}{1+\|y\|^2} - \chi(y)y\Big)N(X_s, dy)\Bigg] \cdot e_j \\ &+ \int_{y\ne 0}\big(y\cdot e_j\chi(y) - y\cdot e_j\big)\chi(y)N(X_s, dy)\Big|\, ds\Bigg) \ . \end{aligned}$$

For all $j = 1, \ldots, n$ and the unit directions e_j

$$\begin{aligned}
&\left| \ell(X_s) \cdot \frac{e_j}{2R} + \int_{y \neq 0} \left[\left(\frac{y \cdot \frac{e_j}{2R}}{1 + \|y\|^2} - \chi(y) y \cdot \frac{e_j}{2R} \right) \right.\right. \\
&\qquad\qquad \left.\left. + \; y \cdot \frac{e_j}{2R} (\chi(y) - 1) \chi(y) \right] N(X_s, dy) \right| \\
&\qquad \leq \left| \ell(X_s) \cdot \frac{e_j}{2R} + \int_{y \neq 0} \left(\frac{y \cdot \frac{e_j}{2R}}{1 + \|y\|^2} - \sin \frac{y \cdot e_j}{2R} \right) N(X_s, dy) \right| \\
&\qquad\quad + \left| \int_{y \neq 0} \left(\sin \frac{y \cdot e_j}{2R} - \frac{y \cdot e_j}{2R} \right) \chi(y) N(X_s, dy) \right| \\
&\qquad\quad + \left| \int_{y \neq 0} \sin \frac{y \cdot e_j}{2R} (1 - \chi(y)) N(X_s, dy) \right| \\
&\qquad \leq \left| \operatorname{Im} p\Big(X_s, \frac{e_j}{2R}\Big) \right| + 6 \int_{y \neq 0} \frac{(y \cdot e_j)^2 / (2R)^2}{1 + (y \cdot e_j)^2 / (2R)^2} N(X_s, dy) \; .
\end{aligned}$$

In the last step, we used the relations

$$\begin{aligned}
&|\sin \alpha - \alpha| \leq (1 - \cos \alpha) \leq \frac{\alpha^2}{1 + \alpha^2}, \quad |\alpha| \leq 1, \quad \text{and} \\
&1 \leq 5 \frac{\alpha^2}{1 + \alpha^2}, \quad |\alpha| > \frac{1}{2} \; ,
\end{aligned}$$

and observed that $\mathbf{1}_{B_R(0)} \leq \chi = \chi_R \leq \mathbf{1}_{B_{2R}(0)}$.

As in Lemma 2.1, (2.5), (2.6) we get

$$\begin{aligned}
&\mathbb{P}^x \Big((X^\sigma_\bullet - x)^*_t \geq 2R, \tau_R > t \Big) \\
&\quad \leq cnt \sup_{\substack{\|y - x\| \leq S \\ \|\epsilon\| \leq 1}} \left(\left| p\Big(y, \frac{\epsilon}{2R}\Big) \right| + \int_{-\infty}^{\infty} \operatorname{Re} p\Big(y, \frac{\rho\epsilon}{2R}\Big) g(\rho) \, d\rho \right) \; .
\end{aligned}$$

and patching together this estimate with the inequalities (6.5) and (6.4) proves our claim. □

Corollary 6.2 *In the situation of Lemma* 6.1 *we have*

$$\begin{aligned}
&\mathbb{P}^x \big((X_\bullet - x)^*_t \geq 2R \big) \\
&\quad \leq cnt \sup_{\|y - x\| \leq 3R} \sup_{\|\epsilon\| \leq 1} \left(\int_{-\infty}^{\infty} \operatorname{Re} p\Big(y, \frac{\rho\epsilon}{2R}\Big) g(\rho) \, d\rho + \left| p\Big(y, \frac{\epsilon}{2R}\Big) \right| \right)
\end{aligned}$$

Proof. Choose $S = 3R$ in Lemma 6.1 and observe that for $\sigma = \sigma^x_{3R}$ of (6.2) we have

$$\big\{ (X^\sigma_\bullet - x)^*_t \geq 2R \big\} = \big\{ (X_\bullet - x)^*_t \geq 2R \big\} \; .$$

Since $\left(X^\sigma_\bullet - x\right)^*_t \leq (X_\bullet - x)^*_t$, the inclusion "$\subset$" is obvious. Conversely, assume that $\omega_0 \in \left\{(X_\bullet - x)^*_t \geq 2R\right\}$. Then for every $\delta > 0$ there is a $s_0 = s_0(\omega_0, \delta)$ such that $\|X_{s_0}(\omega_0) - x\| > 2R - \delta$. If $s_0 \leq \sigma(\omega_0)$, $\|X^\sigma_{s_0}(\omega_0) - x\| = \|X_{s_0}(\omega_0) - x\| > 2R - \delta$; if $s_0 \geq \sigma(\omega_0)$, $\|X^\sigma_{s_0}(\omega_0) - x\| = \|X_\sigma(\omega_0) - x\| \geq 3R$, and – δ being arbitrary – in both cases $\omega_0 \in \left\{\left(X^\sigma_\bullet - x\right)^*_t \geq 2R\right\}$.

The assertion follows now directly from Lemma 6.1. □

In order to show (4.4) and (4.5), we have to assume that for some absolute constant c_0

$$|\operatorname{Im} p(x, \xi)| \leq c_0 \operatorname{Re} p(x, \xi) \quad \text{for all} \quad x, \xi \in \mathbb{R}^n \tag{6.9}$$

is satisfied. Clearly, we may – and will! – assume that $c_0 > 1$.

Lemma 6.3 *Let* $(A, D(A))$ *be a Feller generator such that* $C^\infty_c(\mathbb{R}^n) \subset D(A)$ *and* $A|C^\infty_c(\mathbb{R}^n) = -p(x, D)$. *Assume that the symbol is given by* (4.1), *has Lévy characteristics* $(0, \ell(x), Q(x), N(x, dy))$, *and satisfies* $\|p(\bullet, \xi)\|_\infty \leq c(1 + \|\xi\|^2)$ *and* (6.9). *Then we have for all* $R > 0$

$$\mathbb{P}^x\left((X_\bullet - x)^*_t < R\right) \leq \frac{2}{\cos\frac{1}{4\kappa}} \; \frac{1}{t \inf_{\|x-y\| \leq R} \sup_{\|\epsilon\| \leq 1} \; \operatorname{Re} p\left(y, \frac{\epsilon}{4\kappa R}\right)}$$

where the constant $\kappa = (4 \arctan\frac{1}{2c_0})^{-1}$ *depends only on the constant* c_0 *of* (6.9).

Proof. Let $\sigma = \sigma_R$ be the first exit time from $\overline{B_R(x)}$ given by (6.2). Then

$$\left\{\sup_{s \leq t} \|X_s - x\| < R\right\} \subset \{\sigma > t\} \subset \left\{\sup_{s \leq t} \|X_s - x\| \leq R\right\} , \tag{6.10}$$

and $\|\Delta X_s\| \leq 2R$ whenever $s < \sigma$. In particular, $\sigma \leq \tau_{2R}$, if $\tau_{2R} := \inf\{t > 0 : \|\Delta X_t\| > 2R\}$ denotes the time of the first jump $> 2R$. Thus we find for an arbitrary $\epsilon \in \mathbb{R}^n$ with $\|\epsilon\| \leq 1$

$$\begin{aligned}
\mathbb{P}^x\left((X_\bullet - x)^*_t < R\right) &= \mathbb{P}^x\left((X_\bullet - x)^*_t < R, \tau_{2R} > t, \sigma > t\right) \\
&\leq \mathbb{P}^x\left(\|X_\bullet - x\|_t < R, \tau_{2R} > t, \sigma > t\right) \\
&\leq \mathbb{P}^x\left(\cos\frac{(X_t - x)\cdot\epsilon}{4\kappa R} \geq \cos\frac{1}{4\kappa}, \tau_{2R} > t, \sigma > t\right) .
\end{aligned}$$

Here we used that cos is decreasing in $[0, \pi/4)$ and that on the set $\{\sigma > t\}$

$$\frac{(X_t - x)\cdot\epsilon}{4\kappa R} \leq \frac{\|X_t - x\|}{4\kappa R} \leq \frac{R}{4\kappa R} = \frac{1}{4\kappa} = \arctan\frac{1}{2c_0} \leq \frac{1}{2} < \frac{\pi}{4} .$$

This yields

$$\begin{aligned}\mathbb{P}^x\big((X_\bullet - x)_t^* < R\big) &\le \mathbb{P}^x\left(\cos\frac{(X_t - x)\cdot\epsilon}{4\kappa R} \ge \cos\frac{1}{4\kappa}, \sigma > t\right)\\ &= \mathbb{P}^x\left(\cos\frac{(X_t^\sigma - x)\cdot\epsilon}{4\kappa R} \ge \cos\frac{1}{4\kappa}, \sigma > t\right)\\ &\le \frac{1}{\cos\frac{1}{4\kappa}}\mathbb{E}^x\left(\cos\frac{(X_t^\sigma - x)\cdot\epsilon}{4\kappa R}\right) .\end{aligned}$$

Let $I(p)$ denote the integro-differential representation for the generator $\widetilde{p(x,D)}$, see (2.7). Corollary 3.6 and a standard optional stopping argument imply

$$\begin{aligned}&\mathbb{P}^x\big((X_\bullet - x)_t^* < R\big)\\ &\quad\le \frac{1}{\cos\frac{1}{4\kappa}}\left(1 + \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\left[I_z(p)\left(\cos\frac{(z-x)\cdot\epsilon}{4\kappa R}\right)\right]_{z=X_s} ds\right)\right)\\ &\quad\le \frac{1}{\cos\frac{1}{4\kappa}}\left(1 - \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\ \mathrm{Re}\left[\exp\left(\frac{\mathrm{i}(X_s - x)\cdot\epsilon}{4\kappa R}\right)p\left(X_s,\frac{\epsilon}{4\kappa R}\right)\right]ds\right)\right)\\ &\quad= \frac{1}{\cos\frac{1}{4\kappa}}\left(1 - \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\cos\left(\frac{(X_s - x)\cdot\epsilon}{4\kappa R}\right)\mathrm{Re}\,p\left(X_s,\frac{\epsilon}{4\kappa R}\right)ds\right)\right.\\ &\qquad\left.+\mathbb{E}^x\left(\int_0^{t\wedge\sigma}\sin\left(\frac{(X_s - x)\cdot\epsilon}{4\kappa R}\right)\mathrm{Im}\,p\left(X_s,\frac{\epsilon}{4\kappa R}\right)ds\right)\right) .\end{aligned}$$

(We skipped the simple but otherwise tedious calculation $\mathrm{e}^{-\mathrm{i}z\cdot\xi}I(p)(\mathrm{e}^{\mathrm{i}\bullet\xi})(z) = -p(z,\xi)$). By (6.9) and since $\frac{1}{4\kappa} < \frac{\pi}{4}$ we get

$$\begin{aligned}&\left|\mathbb{E}^x\left(\int_0^{t\wedge\sigma}\sin\left(\frac{(X_s - x)\cdot\epsilon}{4\kappa R}\right)\mathrm{Im}\,p\left(X_s,\frac{\epsilon}{4\kappa R}\right)ds\right)\right|\\ &\qquad\le \mathbb{E}^x\left(\int_0^{t\wedge\sigma}c_0\left|\sin\left(\frac{(X_s - x)\cdot\epsilon}{4\kappa R}\right)\right|\ \mathrm{Re}\,p\left(X_s,\frac{\epsilon}{4\kappa R}\right)ds\right) ,\end{aligned}$$

but for $s < t\wedge\sigma$ we have $\|X_s - x\| \le R$ and therefore

$$\begin{aligned}\cos\frac{(X_s - x)\cdot\epsilon}{4\kappa R} - c_0\left|\sin\frac{(X_s - x)\cdot\epsilon}{4\kappa R}\right| &\ge \cos\frac{1}{4\kappa} - c_0\sin\frac{1}{4\kappa}\\ &= \cos\frac{1}{4\kappa} - \frac{1}{2}\cot\frac{1}{4\kappa}\sin\frac{1}{4\kappa}\\ &= \frac{1}{2}\cos\frac{1}{4\kappa} .\end{aligned}$$

We have thus proved

$$\begin{aligned}
&\mathbb{P}^x\big((X_\bullet - x)^*_t < R\big)\\
&\le \frac{1}{\cos\frac{1}{4\kappa}}\left(1 - \frac{1}{2}\cos\frac{1}{4\kappa}\,\mathbb{E}^x\left(\int_0^{t\wedge\sigma} \operatorname{Re} p\Big(X_s, \frac{\epsilon}{4\kappa R}\Big)\,ds\right)\right)\\
&\le \frac{1}{\cos\frac{1}{4\kappa}}\left(1 - \frac{1}{2}\cos\frac{1}{4\kappa}\,\mathbb{E}^x\left(\int_0^{t} \mathbf{1}_{\{t\le\sigma\}} \operatorname{Re} p\Big(X_s, \frac{\epsilon}{4\kappa R}\Big)\,ds\right)\right)\\
&\le \frac{1}{\cos\frac{1}{4\kappa}}\left(1 - \frac{t}{2}\cos\frac{1}{4\kappa}\inf_{\|y-x\|\le R} \operatorname{Re} p\Big(y, \frac{\epsilon}{4\kappa R}\Big)\mathbb{P}^x(\sigma \ge t)\right) .
\end{aligned}$$

The last estimate required (6.10). Since

$$1 - \alpha \le \alpha^{-N} \quad \text{for all} \quad \alpha \ge 0, N \in \mathbb{N}$$

holds, we get

$$\mathbb{P}^x\big((X_\bullet - x)^*_t < R\big) \le \left(\frac{1}{\cos\frac{1}{4\kappa}}\right)^{N+1}\left(\frac{2}{t\inf_{\|y-x\|\le R}\operatorname{Re} p\big(y,\frac{\epsilon}{4\kappa R}\big)\mathbb{P}^x(\sigma\ge t)}\right)^N ,$$

and a further application of (6.10) yields

$$\big[\mathbb{P}^x\big((X_\bullet - x)^*_t < R\big)\big]^{N+1} \le \frac{2^N/\big(\cos\frac{1}{4\kappa}\big)^{N+1}}{\big(t\inf_{\|y-x\|\le R}\ \operatorname{Re} p\big(y,\frac{\epsilon}{4\kappa R}\big)\big)^N} .$$

Taking roots and letting $N \to \infty$ finally shows

$$\begin{aligned}
\mathbb{P}^x\big((X_\bullet - x)^*_t < R\big) &\le \frac{2^{\frac{N}{N+1}}}{\cos\frac{1}{4\kappa}}\,t^{-\frac{N}{N+1}}\left(\inf_{\|y-x\|\le R}\ \operatorname{Re} p\Big(y, \frac{\epsilon}{4\kappa R}\Big)\right)^{-\frac{N}{N+1}}\\
&\to \frac{2}{\cos\frac{1}{4\kappa}}\,t^{-1}\left(\inf_{\|y-x\|\le R}\ \operatorname{Re} p\Big(y, \frac{\epsilon}{4\kappa R}\Big)\right)^{-1} .
\end{aligned}$$

This finishes the proof, since $\epsilon \in \mathbb{R}^n$, $\|\epsilon\| \le 1$, was arbitrarily chosen. □

Corollary 6.4 *In the situation of Lemma* 6.3 *one has also*

$$\mathbb{P}^x\big((X_\bullet - x)^*_t < R/2\big) \le \frac{16}{\cos^2\frac{1}{4\kappa}}\left(\frac{1}{t\inf_{\|x-y\|\le 3R/2}\sup_{\|\epsilon\|\le 1}\ \operatorname{Re} p\big(y,\frac{1}{4\kappa R}\big)}\right)^2$$

Proof. The key to the modified estimate is the inclusion

$$\left\{\sup_{s\le t}\|X_s - x\| < R/2\right\}$$
$$\subset \left\{\sup_{s\le t/2}\|X_s - x\| < R/2\right\} \cap \left\{\sup_{s\le t/2}\|X_{s+t/2} - X_{t/2}\| < R\right\}$$

and an application of the Markov property

$$\begin{aligned}
&\mathbb{P}^x\big((X_\bullet - x)^*_t < R/2\big)\\
&\le \mathbb{E}^x\Big(\mathbf{1}_{\{(X_\bullet - x)^*_{t/2} < R/2\}}\mathbf{1}_{\{\sup_{s\le t/2}\|X_{s+t/2}-X_{t/2}\|<R\}}\Big)\\
&= \mathbb{E}^x\Big(\mathbf{1}_{\{(X_\bullet - x)^*_{t/2} < R/2\}}\mathbb{E}^{X_{t/2}}\Big(\mathbf{1}_{\{(X_\bullet - X_0)^*_{t/2} < R\}}\Big)\Big)\\
&\le \left[\sup_{\|y-x\|\le R/2}\mathbb{E}^y\Big(\mathbf{1}_{\{(X_\bullet - y)^*_{t/2} < R\}}\Big)\right]\mathbb{E}^x\Big(\mathbf{1}_{\{(X_\bullet - x)^*_{t/2} < R/2\}}\Big)\\
&\le \left[\sup_{\|y-x\|\le R/2}\mathbb{E}^y\Big(\mathbf{1}_{\{(X_\bullet - y)^*_{t/2} < R\}}\Big)\right]^2\\
&\le \frac{16}{\cos^2\frac{1}{4\kappa}}t^{-2}\sup_{\|y-x\|\le R/2}\left(\frac{1}{\inf_{\|y-z\|\le R}\sup_{\|\epsilon\|\le 1}\operatorname{Re}p\big(z,\frac{\epsilon}{4\kappa R}\big)}\right)^2
\end{aligned}$$

where we used Lemma 6.3. The assertion now follows from the triangle inequality.

Lemma 4.1 is now easily derived: (4.3) and (4.4) are direct consequences of Corollary 6.2 and Lemma 6.3, respectively, and (4.5) follows from Corollary 6.4.

7 References

[1] Bass, R.F.: Uniqueness in Law for Pure Jump Type Markov Processes. Probab. Theor. Rel. Fields. **79**, 271–287 (1988)

[2] Berg, C., Forst, G.: Non-symmetric translation invariant Dirichlet forms. Invent. Math. **21**, 199–212 (1973)

[3] Berg, C., Forst, G.: Potential Theory on Locally Compact Abelian Groups. Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete. II. Ser. Bd. **87**, Berlin 1975

[4] Blumenthal, R. M., Getoor, R. K.: Sample Functions of Stochastic Processes with Stationary Independent Increments. J. Math. Mech. **10**, 493–516 (1961)

[5] Bony, J.-M., Courrège, Ph. et, Priouret, P.: Semi-groupes de Feller sur une variété à bord compacte et problème aux limites intégro-différentiels du second

ordre donnant lieu au principe du maximum. Ann. Inst. Fourier, Grenoble **18**.2, 369–521 (1968)

[6] Courrège, Ph.: Sur la forme intégro-différentielle des opérateurs de C_K^∞ *dans C* satisfaisant au principe du maximum. Sém. Théorie du Potentiel (1965/66) exposé 2, 38 pp.

[7] Ethier, St., Kurtz, E. Th. G.: Markov Processes: Characterization and Convergence. Wiley, Series in Probab. and Math. Stat., New York 1986.

[8] Hoh, W.: Some commutator estimates for pseudo differential operators with negative definite functions as symbols. Integr. Equat. Oper. Th. **17**, 46–53 (1993)

[9] Hoh, W.: Pseudo differential operators with negative definite symbols and the martingale problem. Stochastics and Stochastics Reports **55**, 225–252 (1995)

[10] Hoh, W.: A symbolic calculus for pseudo differential operators generating Feller semigroups. To appear in Osaka J. Math.

[11] Hörmander, L.: The Analysis of Linear Partial Differential Operators II. Springer, Grundlehren Math. Wiss. Bd. **257**, Berlin 1983

[12] Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland Math. Library vol. **24**, North-Holland, Amsterdam 1989 (2nd ed.)

[13] Jacob, N.: Feller semigroups, Dirichlet forms and pseudo differential operators. Forum Math. **4**, 433–446 (1992)

[14] Jacob, N.: A class of Feller semigroups generated by pseudo differential operators. Math. Z. **215**, 151–166 (1994)

[15] Jacob, N.: Non-local (semi-)Dirichlet forms generated by pseudo differential operators, in: Ma, Z.M., Röckner, M.J., Yan A., (eds.), Dirichlet Forms and Stochastic Processes. Proc. Intnl. Conf. Dirichlet Forms and Stochastic Processes. Beijing 1993, de Gruyter, Berlin 1995, 223–233

[16] Jacob, N.: Pseudo-differential operators and Markov processes. Akademie-Verlag, Mathematical Research vol. **94**, Berlin 1996

[17] Jacob, N.: Characteristic functions and symbols in the theory of Feller processes. Potential Analysis **8**, 61–68 (1998)

[18] Jacob, N.: Fourier Analysis, Feller Semigroups, Dirichlet Forms, and Markov Processes. book in preparation

[19] Jacob, N., Leopold, H.-G.: Pseudo differential operators with variable order of differentiation generating Feller semigroups. Integr. Equat. Oper. Th. **17**, 544–553 (1993)

[20] Jacob, N., Schilling, R. L.: An analytic proof of the Lévy-Khinchin formula on $\mathbb{R}^n$. Publ. Math. Debrecen **53**, 69–89 (1998)

[21] Jacod, J., Shiryaev, A. N.: Limit Theorems for Stochastic Processes. Springer, Grundlehren math. Wiss. vol. **288**, Berlin 1987

[22] Kikuchi, K., Negoro, A.: On Markov processes generated by pseudodifferential operators of variable order. Osaka J. Math. **34**, 319–335 (1997)

[23] Komatsu, T.: Pseudo-differential operators and Markov processes. *J. Math. Soc. Japan* **36**, 387–418 (1984)

[24] Kumano-go, H.: Pseudo-Differential Operators. MIT-Press, Cambridge, MA 1981

[25] Negoro, A.: Stable-like processes: construction of the transition density and the behavior of sample paths near $t = 0$, Osaka J. Math. **31**, 189–214 (1994)

[26] Protter, Ph.: Stochastic Integration and Differential Equations. Springer, Appl. Math. vol. **21**, Berlin 1990

[27] Pruitt, W. E.: The growth of random walks and Lévy processes. Ann. Probab. **9**, 948–956 (1981)
[28] Pruitt, W. E., Taylor, S. J.: Packing and covering indices for a general Lévy process. Ann. Probab. **24**, 971–986 (1996)
[29] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Springer, Grundlehren math. Wiss. Bd. **293**, Berlin 1994 (2nd ed.)
[30] Schilling, R. L.: On Feller processes with sample paths in Besov spaces. Math. Ann. **309**, 663–675 (1997)
[31] Schilling, R. L.: Conservativeness of semigroups generated by pseudo differential operators. Potential Analysis **9**, 91–104 (1998)
[32] Schilling, R. L.: Feller processes generated by pseudo-differential operators: On the Hausdorff dimension of their sample paths. J. Theor. Probab. **11**, 303–330 (1998)
[33] Schilling, R. L.: Conservativeness and extensions of Feller semigroups. Positivity **2**, 239–256 (1998)
[34] Stroock, D.: Diffusion processes associated with Lévy generators. Z. Wahrscheinlichkeitstheorie verw. Gebiete **32**, 209–244 (1975)
[35] Tsuchiya, M.: Lévy measure with generalized polar decomposition and the associated SDE with jumps. Stochastics and Stochastics Reports **38**, 95–117 (1992)
[36] von Waldenfels, W.: Fast positive Operatoren. Z. Wahrscheinlichkeitstheorie verw. Geb. **4**, 159–174 (1965)