Option Pricing
Chapter 12 - Local volatility models -

Stefan Ankirchner

University of Bonn

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The volatility surface
Local volatility
Dupire’s formula
Practical note: how to calibrate the loc. volatility function

Further reading:
Implied vola

Implied vola varies with

- different strikes
- different maturities

Recall the volatility smile:
Volatility surface

Volatility surface = graph of the implied volatility as a function of \( t2m \) and moneyness

Data: Exxon Mobile call prices, CBO on 5/12/2010.
Definition of local volatility

One way to make model prices consistent with market prices for Plain Vanilla Options is to assume that the volatility is a function of time and the underlying price:

\[ dS_t = rS_t dt + S_t \sigma(t, S_t) dW_t^Q. \]  \hspace{1cm} (1)

**Definition:** the mapping \((t, x) \mapsto \sigma(t, x)\) is called local volatility function.

**We will see that**

- European call / put prices uniquely determine the local volatility function,
- our pricing PDEs for American and exotic options remain almost the same: only the constant vola has to be replaced with the local volatility function.
Local volatility and transition densities

Proof of Dupire’s formula

Local volatility is determined by European Call Prices

Suppose that the market call prices $C(T, K)$ are known for all possible expiration dates $T > 0$ and strike prices $K \geq 0$. Then the local volatility function is given by

$$\sigma(T, K) = \sqrt{2 \frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K} \frac{K^2 \frac{\partial^2 C}{\partial K^2}}{K^2 \frac{\partial^2 C}{\partial K^2}}}$$

Equation (2) is called **Dupire’s formula**. For the proof we use the transition density of the price process $S_t$. 
Let $\phi(0, x; T, y)$ be the probability density of $S_T$ at $y$ conditional to $S_0 = x$. This means that for any nice $B \subset \mathbb{R}$

$$P_{0,x}(S_T \in B) = \int_B \phi(0, x; T, y) \, dy.$$ 

**Definition:** $\phi(0, x; T, y)$ is called transition density of the process $S_t$.

**Notation:** In the following we fix the initial condition $(0, x)$ and simply write $\phi(T, y) = \phi(0, x; T, y)$. 
Theorem (Kolmogorov forward equation)

Suppose that $S_t$ satisfies

$$dS_t = rS_t dt + S_t \sigma(t, S_t) dW^Q_t, \quad S_0 = x.$$ 

Then the transition density $\phi(T, y)$ of $S_t$ satisfies the PDE

$$\frac{\partial}{\partial T} \phi(T, y) = -\frac{\partial}{\partial y} (ry \phi(T, y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (y^2 \sigma^2(T, y) \phi(T, y)).$$

Remark: The Kolmogorov forward equation is sometimes also called Fokker-Planck equation.
The price of a call option expiring at $T$ and struck at $K$ satisfies
\[
C(T, K) = e^{-rT} E^Q \left[ (S_T - K)^+ \right] \\
= e^{-rT} \int_{\mathbb{R}} (y - K)^+ \phi(T, y) dy \\
= e^{-rT} \int_{K}^{\infty} (y - K) \phi(T, y) dy.
\]
Partial derivatives

\[ C(T, K) = e^{-rT} \int_{K}^{\infty} (y - K) \phi(T, y) dy \]

Observe that

\[ \frac{\partial}{\partial T} C(T, K) = -rC(T, K) + e^{-rT} \int_{K}^{\infty} (y - K) \frac{\partial}{\partial T} \phi(T, y) dy \] (3)

\[ \frac{\partial}{\partial K} C(T, K) = -e^{-rT} \int_{K}^{\infty} \phi(T, y) dy \] (4)

\[ \frac{\partial^2}{\partial K^2} C(T, K) = e^{-rT} \phi(T, K) \] (5)

Proof.
Proof of Dupire’s formula

With the Kolmogorov forward equation we get

\[
\frac{\partial}{\partial T} C(T, K) = -r C(T, K) + e^{-rT} \int_{\infty}^{\infty} (y - K) \frac{\partial}{\partial T} \phi(T, y) dy
\]

\[
= -r C(T, K) - e^{-rT} \int_{K}^{\infty} (y - K) \left[ \frac{\partial}{\partial y} (ry \phi(T, y)) \right. \\
\left. - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( y^2 \sigma^2(T, y) \phi(T, y) \right) \right] dy
\]  

(6)
Suppose that

A1) \( \lim_{y \to \infty} (y - K) \frac{\partial}{\partial y} (y^2 \sigma^2(T, y) \phi(T, y)) = 0, \)

A2) \( \lim_{y \to \infty} (y^2 \sigma^2(T, y) \phi(T, y)) = 0, \)

A3) \( \lim_{y \to \infty} (y - K) ry \phi(T, y) = 0. \)

Then we can show

Dupire’s formula:

\[
\sigma(T, K) = \sqrt{2 \frac{\partial^2 C}{\partial T \partial K} + rK \frac{\partial C}{\partial K}}
\]

Proof.
Question: How to price American or exotic options that are not actively traded?

- Derive the local volatility function from standard European options,
- use the local vola function in the pricing PDE for the American or exotic option considered,
- and solve the PDE numerically.
Example: Pricing D&O call with a local volatility model

Assume that

$$dS_t = rS_t dt + S_t \sigma(t, S_t) dW^Q_t, \quad S_0 = x.$$ 

and that $\sigma(t, x)$ has been calibrated from market prices of standard calls or puts.

Let $v(t, x)$ be the time $t$ D&O call value under the assumption that it has not been knocked out before $t$ and that $S_t = x$. Then $v(t, x)$ satisfies the Black-Scholes PDE

$$v_t(t, x) + r x v_x(t, x) + \frac{1}{2} \sigma^2(t, x) x^2 v_{xx}(t, x) - rv(t, x) = 0,$$

with boundary conditions

$$v(t, L) = 0, \quad 0 \leq t \leq T,$$

$$v(T, x) = (x - K)^+, \quad x > L.$$
How to find the local vola function?

Dupire’s formula requires market prices for all strikes and maturities!
In reality we have market prices for only a finite number of Plain Vanilla Calls.

Idea: Parameterize the local vola function and choose the parameters such that the model prices for European calls & puts are as close as possible to the real market prices.
A simple time-space parameterization of the local vola function is given by

$$\sigma(t, y) = a_0 + a_1 y + a_2 y^2 + a_3 t + a_4 t^2 + a_5 ty$$

Let

- $T_1 \leq \cdots \leq T_m$ be the maturities of traded calls.
- $K_{i,1}, \ldots, K_{i,n(i)}$ strikes of traded calls expiring at $T_i$, $1 \leq i \leq m$.
- $C(T_i, K_{i,j}) = \text{market price of the traded call with expiration } T_i \text{ and strike } K_{i,j}$. 
A simple time-space parameterization cont’d

Assume that the local vola function satisfies

\[ \sigma(t, y) = a_0 + a_1 y + a_2 y^2 + a_3 t + a_4 t^2 + a_5 t y. \]

For a given set of parameters \((a_0, \ldots, a_5)\) one can calculate the model call prices

\[ \hat{C}_{i,j} = e^{-rT_i} \int_{K_{i,j}}^{\infty} (y - K_{i,j}) \phi(0, x; T_i, y) dy. \]

for all \(1 \leq i \leq m\) and \(1 \leq j \leq n(i)\).

The quadratic distance to the market prices is given by

\[ \text{error}(a_0, a_1, a_2, a_3, a_4, a_5) = \sum_{i=1}^{m} \sum_{j=1}^{n(i)} (C(T_i, K_{i,j}) - \hat{C}_{i,j})^2 \]
A simple time-space parameterization cont’d

With a search algorithm one can find \((a_0^*, \ldots, a_5^*)\) such that

\[
\text{error}(a_0^*, \ldots, a_5^*) \approx \min_{(a_0, \ldots, a_5)} \text{error}(a_0, \ldots, a_5).
\]

The local volatility function is then approximately given by

\[
\sigma(t, y) = a_0^* + a_1^* y + a_2^* y^2 + a_3^* t + a_4^* t^2 + a_5^* t y.
\]
BS implied vola and local volatility

**Notation:** $\sigma_{BS}(T, K) = \text{BS implied vola for a call with strike } K \text{ and exp. date } T$

Note that the market call price $C(T, K)$ satisfies

$$C(T, K) = \text{BS\_call}(S_0, K, T, \sigma_{BS}(T, K), r).$$

We will show that the local volatility function $\sigma(T, K)$ is uniquely determined by the BS implied volatility function $\sigma_{BS}(T, K)$.

Therefore: instead of parameterizing and calibrating the local vola function directly, one can proceed as follows:

- Parameterize the BS implied vola function
- Calibrate the BS implied vola function to market prices
- Calculate the *local* vola function from the calibrated BS implied vola function.
Linking local vola and BS implied vola

Some new variables:

- log moneyness $y = \log \left( \frac{K}{e^{rT}S_0} \right)$
- implied total variance $w(T, y) = T \sigma_{BS}^2(T, e^{rT}S_0e^y)$

**Lemma**

The local volatility function satisfies

$$\sigma^2(T, e^{rT}S_0e^y) = \frac{\partial w}{\partial T}(T, y)$$

$$= 1 - \frac{y}{w} \frac{\partial w}{\partial y}(T, y) + \frac{1}{2} \frac{\partial^2 w}{\partial y^2}(T, y) + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w}\right) \left(\frac{\partial w}{\partial y}\right)^2(T, y)$$
Steps in the proof of Equation (7)

a) First write the BS call price as a function of log moneyness and implied total variance

$$C^{BS}(y, w) := \text{BS\_call}(S_0, e^{rT} S_0 e^{y}, T, \sqrt{\frac{w}{T}}, r).$$

Observe that

$$C^{BS}(y, w) = S_0 \left[ \Phi(-\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}) - e^{y} \Phi(-\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}) \right]$$

The partial derivatives of $C^{BS}$ satisfy

$$\frac{\partial^2 C^{BS}}{\partial w^2}(y, w) =$$

$$\frac{\partial^2 C^{BS}}{\partial y \partial w}(y, w) =$$

$$\frac{\partial^2 C^{BS}}{\partial y^2}(y, w) - \frac{\partial C^{BS}}{\partial y}(y, w) =$$
Steps in the proof of Equation (7)

b) Write the market call price as a function of $T$ and log moneyness:

$$\tilde{C}(T, y) := C(T, e^{rT}S_0e^y).$$

Dupire’s formula implies

$$\frac{1}{2} \sigma^2(T, e^{rT}S_0e^y)(\frac{\partial^2 \tilde{C}}{\partial y^2} - \frac{\partial \tilde{C}}{\partial y})(T, y) = \tilde{C}_T(T, y).$$  \hspace{1cm} (8)

Proof of (8).
c) Observe that by definition we have

\[ \tilde{\mathcal{C}}(T, y) = C^{BS}(y, w(y, T)). \]

With this one can rewrite Dupire’s formula in terms of \( C^{BS} \) and then derive (7).

**Proof.**
Objections to local volatility models

Local volatility models are criticized because:

- bad statistical properties:
  - the local volatility functions change considerably over time.
    - E.g. parameters usually change from one week to the next
  - bad prediction properties


- The local volatility model predicts the smile/skew to move in the opposite direction as the underlying; in reality, both move in the same direction.


- hedging based on volatility models is inconsistent
- ad hoc model; no economic explanation of the local volatility function