Existence of solutions for nonlinear fractional stochastic differential equations

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\textbf{A B S T R A C T}

The fractional stochastic differential equations have wide applications in various fields of science and engineering. This paper addresses the issue of existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces. Using fractional calculations, fixed point technique, stochastic analysis theory and methods adopted directly from deterministic fractional equations, new set of sufficient conditions are formulated and proved for the existence of mild solutions for the fractional impulsive stochastic differential equation with infinite delay. Further, we study the existence of solutions for fractional stochastic semilinear differential equations with nonlocal conditions. Examples are provided to illustrate the obtained theory.

\section{1. Introduction}

Fractional dynamical equations have played a central role in the modeling of anomalous relaxation and diffusion processes. The fact that fractional derivatives introduce a convolution integral with a power-law memory kernel makes the fractional differential equations an important one to describe memory effects in complex systems [1]. The increasing interest of fractional differential equations is motivated by their applications in various fields of science such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering [2–5]. Hilfer [6,7] showed that time fractional derivatives are equivalent to infinitesimal generators of generalized time fractional evolutions that arise in the transition from microscopic to macroscopic time scales. Also, it is shown that this transition from the ordinary time derivative to the fractional time derivative arises in different physical problems [8]. Further, many different applications of fractional calculus are presented in [1]. By considering some special cases of the fractional generalized Langevin equation, a new class of closed analytic expressions for the mean square displacement of single file diffusion has been obtained in [9], and also Gaussian models of retarded and accelerated anomalous diffusion are considered in [10]. Sandev et al. [11] studied analytically a generalized fractional Langevin equation and also general formulas for calculation of variances and the mean square displacement are derived. Tomovski [12] proved the existence and uniqueness of solutions to Cauchy type problems for fractional differential equations with composite fractional derivative operator on a finite interval of the real axis in spaces of summable functions.
On the other hand, the stochastic differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations, (see [13–16] and the references therein). Chang et al. [17] investigated the existence of square-mean almost automorphic mild solutions for a stochastic differential equation in a real separable Hilbert space by applying a composition theorem together with Schauder’s fixed point theorem. Ren and Sun [18] studied the existence of solutions for a class of second-order neutral stochastic evolution equations with infinite delay in which the initial value belongs to the abstract space. The existence and uniqueness of quadratic mean almost periodic mild solutions for a class of stochastic differential equations in a real separable Hilbert space by employing the contraction mapping principle and an analytic semigroup of linear operators has been discussed [19]. However, only few papers deal with the existence result for stochastic fractional systems. Cui and Yan [20] studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii’s fixed point theorem. The existence and uniqueness for a class of fractional stochastic delay and evolution differential equations has been established in [21,22].

In particular, differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as impulsive fractional differential equations and existence results for such equations have been studied by several authors [23,24]. Feckan [5] studied the existence of a solution for a class of impulsive differential equations with fractional derivative. The existence of mild solutions for a class of impulsive fractional partial semilinear differential equations has been discussed in [23]. The existence results for a class of impulsive fractional order semilinear evolution equations with infinite delay has been reported in [25]. Moreover, it is known that the development of the theory of functional differential equations with infinite delay depends on a choice of a phase space. The existence of a mild solution for fractional semilinear differential equations with infinite delay has been discussed in [26]. Wang et al. [27] studied the solvability and optimal controls for a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. The stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences [28,18]. Wei and Wang [28] considered a class of stochastic functional differential equations within finite delay in which the initial value belongs to the phase space. The existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay has been investigated in [29].

Since impulsive effects also widely exist in fractional stochastic differential systems, it is important and necessary to discuss the qualitative properties for stochastic fractional equations with impulsive perturbations and infinite delay. However, to the authors’ knowledge, no result has been reported on the existence problem of impulsive fractional stochastic differential equations with infinite delay and the aim of this paper is to fill this gap.

Motivated by this consideration, in this paper, first we shall discuss the existence of solutions for a class of impulsive fractional stochastic differential equations with infinite delay by using some appropriate fixed point theorems and evolution system theory. Moreover, the nonlocal conditions give a better description in applications than standard ones, the topic of nonlocal problems has been studied extensively for the existence of fractional differential equations [30,31]. As indicated in [32] and references therein, the Cauchy problem with nonlocal initial condition can be applied in physics with better effect than the classical Cauchy problem with traditional initial conditions. Taking this into account, in this paper we also study the existence of mild solutions for semilinear fractional stochastic differential equations with nonlocal conditions.

2. Preliminaries and basic properties

Let $H$, $K$ be two separable Hilbert spaces and $\mathcal{L}(K, H)$ be the space of bounded linear operators from $K$ into $H$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $H$, $K$ and $\mathcal{L}(K, H)$, and use $(\cdot, \cdot)$ to denote the inner product of $H$ and $K$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. $W = (W_t)_{t \geq 0}$ be a $Q$-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the covariance operator $Q$ such that $\text{Tr}Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k \geq 1}$ in $K$, a bounded sequence of nonnegative real numbers $\lambda_k$ such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k \geq 1}$ such that

$$
(w(t), e)_K = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e_k, e \rangle_K \beta_k(t), \quad e \in K, \ t \geq 0.
$$

Let $L_2^0 = \mathcal{L}_2(Q^{1/2}K, \mathbb{H})$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}K$ to $\mathbb{H}$ with the inner product $\langle \varphi, \psi \rangle_{L_2^0} = \text{Tr} [\varphi Q \psi^*]$. We consider the following impulsive fractional stochastic differential equations with infinite delay in the form

$$
\begin{align*}
D_t^\alpha x(t) &= Ax(t) + f(t, x_t, B_1 x(t)) + \sigma(t, x_t, B_2 x(t)) \frac{dw(t)}{dt}, \quad t \in J = [0, T], T > 0, \ t \neq t_k, \\
\Delta x(t_k) &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\
x(t) &= \phi(t), \quad \phi(t) \in B_0,
\end{align*}
$$

(2.1)
where $D^\alpha_t$ is the Caputo fractional derivative of order $\alpha$, $0 < \alpha < 1$; $x(\cdot)$ takes the value in the separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is the infinitesimal generator of an $\alpha$-resolvent family $S_\alpha(t)_{t \geq 0}$. The history $x_t : (-\infty, 0] \rightarrow \mathbb{H}$, $x_t(\theta) = x(t + \theta)$, $\theta \leq 0$, belongs to an abstract phase space $\mathcal{B}_h$; $f : \mathbb{J} \times \mathcal{B}_h \times \mathbb{H} \rightarrow \mathbb{H}$ and $\sigma : \mathbb{J} \times \mathcal{B}_h \times L^2_0 \rightarrow \mathbb{H}$ are appropriate functions to be specified later; $k_k : \mathcal{B}_h \rightarrow H$, $k = 1, 2, \ldots, m$, are appropriate functions. The terms $B_1x(t)$ and $B_2x(t)$ are given by $B_1x(t) = \int_0^t K(t, s)x(s)ds$ and $B_2x(t) = \int_0^t P(t, s)x(s)ds$ respectively, where $K, P \in C(D, \mathbb{R}^+)$ are the set of all positive continuous functions on $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of $w$ with finite second moments.

Now, we present the abstract space phase $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \rightarrow (0, \infty)$ with $l = \int_{-\infty}^0 h(t)dt < \infty$ a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h = \left\{ \phi : (-\infty, 0] \rightarrow \mathbb{H}, \text{ for any } a > 0, \ (E|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable} \right\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (E|\phi(\theta)|^2)^{1/2}ds, \ \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \| \cdot \|_{\mathcal{B}_h})$ is a Banach space [29,18].

Now we consider the space

$$\mathcal{B}_b = \left\{ x : (-\infty, T] \rightarrow \mathbb{H} \text{ such that } x|_{J_k} \in C(J_k, \mathbb{H}) \text{ and there exist} x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), \ x_0 = \phi \in \mathcal{B}_b, \ k = 1, 2, \ldots, m \right\},$$

where $x|_{J_k}$ is the restriction of $x$ to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$. The function $\| \cdot \|_{\mathcal{B}_b}$ to be a seminorm in $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{s \in [0, T]} (E|x(s)|^2)^{1/2}, \ x \in \mathcal{B}_b.$$

We recall the following lemma [29].

**Lemma 2.1.** Assume that $x \in \mathcal{B}_b$; then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,

$$l(E\|x(t)\|^2)^{1/2} \leq l \sup_{s \in [0, t]} (E|x(s)|^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s)ds < \infty$.

A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C e^{\mu z} \mu^{-\alpha} d\mu, \ \alpha, \beta \in \mathbb{C}, \ \Re(\alpha) > 0, \ (2.2)$$

where $C$ is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^\beta$ counter clockwise. For short, $E_{\alpha}(z) = E_{\alpha, 1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_2(z) = e^z$ and the cosine function: $E_2(z^2) = \cos h(z)$, $E_2(-z^2) = \cos z$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha, \beta}(\alpha \omega^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, \ \Re\lambda > \omega^\frac{1}{\alpha}, \ \omega > 0, \ (2.3)$$

and for more details see [4].

**Definition 2.2 ([33]).** A closed and linear operator $A$ is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:

(1) $\rho(A) \subset \sigma(\theta, \omega) = \{ \lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta \}$.

(2) $\|\Re(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \ \lambda \in \sigma(\theta, \omega)$. 

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Definition 2.3 ([2]). Let $A$ be a closed and linear operator with the domain $D(A)$ defined in a Banach space $X$. Let $\rho(A)$ be the resolvent set of $A$. We say that $A$ is the generator of an $\alpha$-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is a Banach space of all bounded linear operators from $X$ into $X$ and the corresponding norm is denoted by $\| \cdot \|$, such that $\{\lambda^\alpha : \text{Re} \lambda > \omega\} \subset \rho(A)$ and

$$ (\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t) x dt, \quad \text{Re} \lambda > \omega, \ x \in X, \tag{2.4} $$

where $S_\alpha(t)$ is called the $\alpha$-resolvent family generated by $A$.

Definition 2.4 ([25]). Let $A$ be a closed linear operator with the domain $D(A)$ defined in a Banach space $X$ and $\alpha > 0$. We say that $A$ is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{\lambda_\alpha : \text{Re} \lambda > \omega\} \subset \rho(A)$ and

$$ \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t) x dt, \quad \text{Re} \lambda > \omega, \ x \in X, \tag{2.5} $$

where $S_\alpha(t)$ is called the solution operator generated by $A$.

The concept of the solution operator is closely related to the concept of a resolvent family [25]. For more details on $\alpha$-resolvent family and solution operators, we refer the reader to [25,34] and the references therein.

Definition 2.5 ([35]). The Caputo derivative of order $\alpha$ for a function $f : [0, \infty) \to \mathbb{R}$, which is at least $n$-times differentiable can be defined as

$$ D_\alpha^tf(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = \int_0^t f^{(n)}(s) ds, \tag{2.6} $$

for $n-1 \leq \alpha < n$, $n \in \mathbb{N}$. If $0 < \alpha \leq 1$, then

$$ D_\alpha^tf(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds. \tag{2.7} $$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$ \mathcal{L}[D_\alpha^tf(t); \lambda] = \lambda^\alpha \mathcal{L}[f(t); \lambda] - \sum_{k=0}^{n-1} \lambda^{\alpha-k} f^{(k)}(0); \quad n-1 \leq \alpha < n. $$

Lemma 2.6 ([25]). If $f$ satisfies the uniform Holder condition with the exponent $\beta \in (0, 1]$ and $A$ is a sectorial operator, then the unique solution of the Cauchy problem

$$ D_\alpha^tx(t) = Ax(t) + f(t, x_t, Bx(t)), \quad t > t_0, \ t_0 \geq 0, \ 0 < \alpha < 1, \tag{2.8} $$

$$ x(t) = \phi(t), \quad t \leq t_0, \tag{2.9} $$

is given by

$$ x(t) = T_\alpha(t-t_0)\phi(t_0) + \int_{t_0}^t S_\alpha(t-s)f(s, x_s, Bx(s))ds, \tag{2.10} $$

where

$$ T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\mathcal{B}_R} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda, \tag{2.11} $$

$$ S_\alpha(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\mathcal{B}_R} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda, \tag{2.12} $$

here $\mathcal{B}_R$ denotes the Bromwich path; $S_\alpha(t)$ is called the $\alpha$-resolvent family and $T_\alpha(t)$ is the solution operator generated by $A$.

Next, we mention the statement of Krasnoselskii’s fixed point theorem [25].

Theorem 2.7. Let $B$ be a nonempty closed convex of a Banach space $(X, \| \cdot \|)$. Suppose that $P$ and $Q$ map $B$ into $X$ such that

(i) $Px + Qy \in B$ whenever $x, y \in B$;
(ii) $P$ is compact and continuous;
(iii) $Q$ is a contraction mapping.

Then there exists $z \in B$ such that $z = Px + Qz$. 

3. Stochastic fractional equations with infinite delay and impulses

In this section, we first establish the existence of mild solutions to nonlinear fractional stochastic equation (2.1). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (2.1) with infinite delay and impulses. In order to establish the results, we impose the following conditions.

(H1) If \( \alpha \in (0, 1) \) and \( A \in \mathcal{A}^{\alpha}(\theta_0, \omega_0) \), then for any \( x \in \mathbb{H} \) and \( t > 0 \) we have \( \| T_a(t) \| \leq M e^{at} \) and \( \| S_a(t) \| \leq C e^{at} (1 + t^{\alpha - 1}) \), \( \omega > \omega_0 \). Thus, we have

\[
\| T_a(t) \| \leq \tilde{M}_T \quad \text{and} \quad \| S_a(t) \| \leq t^{\alpha - 1}\tilde{M}_S,
\]

where \( \tilde{M}_T = \sup_{0 \leq t \leq T} \| T_a(t) \| \), and \( \tilde{M}_S = \sup_{0 \leq t \leq T} C e^{at} (1 + t^{1-\alpha}) \) (for more details, see [23]).

(H2) There exist \( \mu_1, \mu_2 > 0 \) such that

\[
E\| f(t, \gamma, x) - f(t, \psi, y) \|_{\mathcal{H}}^2 \leq \mu_1 \| \gamma - \psi \|_{\mathcal{H}}^2 + \mu_2 E\| x - y \|_{\mathcal{H}}^2.
\]

(H3) There exist \( \nu_1, \nu_2 > 0 \) such that

\[
E\| \sigma(t, \gamma, x) - \sigma(t, \psi, y) \|_{\mathcal{H}}^2 \leq \nu_1 \| \gamma - \psi \|_{\mathcal{H}}^2 + \nu_2 E\| x - y \|_{\mathcal{H}}^2.
\]

(H4) For each \( k = 1, 2, \ldots, m \), there exist \( n_k > 0 \) such that

\[
E\| l_k(x) - l_k(y) \|_{\mathcal{H}}^2 \leq n_k E\| x - y \|_{\mathcal{H}}^2, \quad \text{for all } x, y \in \mathbb{H}.
\]

Now, we present the definition of mild solutions for the system (2.1) based on the paper [25].

**Definition 3.1.** An \( \mathcal{F}_t \)-adapted stochastic process \( x : (-\infty, T] \rightarrow \mathbb{H} \) is called a mild solution of the system (2.1) if \( x_0 = \phi \in \mathcal{B}_0 \) satisfying \( x_0 \in L^2_0(\Omega, \mathbb{H}) \) and the following conditions hold.

(i) \( x(t) \) is \( \mathcal{B}_0 \)-valued and the restriction of \( x(\cdot) \) to the interval \( (t_k, t_{k+1}) \), \( k = 1, 2, \ldots, m \) is continuous.

(ii) for each \( t \in J \), \( x(t) \) satisfies the following integral equation

\[
x(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0], \\
\int_0^t S_a(t-s) f(s, x_s, B_1x(s)) ds \\
+ \int_0^t S_a(t-s) \sigma(s, x_s, B_2x(s)) dw(s), & t \in [0, t_1], \\
T_a(t-t_1)(x(t_1^-) + l_1(x(t_1^-))) + \int_{t_1}^t S_a(t-s) f(s, x_s, B_1x(s)) ds \\
+ \int_{t_1}^t S_a(t-s) \sigma(s, x_s, B_2x(s)) dw(s), & t \in (t_1, t_2], \\
\vdots
\end{cases}
\]

\[
\int_{t_{k-1}}^t S_a(t-s) f(s, x_s, B_1x(s)) ds \\
+ \int_{t_{k-1}}^t S_a(t-s) \sigma(s, x_s, B_2x(s)) dw(s), & t \in (t_{k-1}, t_k], \\
\vdots
\end{cases}
\]

\[
T_a(t-t_m)(x(t_m^-) + l_m(x(t_m^-))) + \int_{t_m}^t S_a(t-s) f(s, x_s, B_1x(s)) ds \\
+ \int_{t_m}^t S_a(t-s) \sigma(s, x_s, B_2x(s)) dw(s), & t \in (t_m, T].
\]

(iii) \( \Delta x|_{t=t_k} = l_k(x(t_k^-)) \), \( k = 1, \ldots, m \) the restriction of \( x(\cdot) \) to the interval \([0, T) \setminus \{ t_1, \ldots, t_m \} \) is continuous.

Now, we are in a position to state the existence theorem. Our first theorem is based on the Banach contraction principle.

**Theorem 3.2.** Assume that the conditions (H1)–(H4) hold. If \( A \in \mathcal{A}^{\alpha}(\theta_0, \omega_0) \), then the system (2.1) has a unique mild solution provided that

\[
\max_{1 \leq i \leq m} \left( 4 \tilde{M}_T^2 (1 + n_i) + 4 \tilde{M}_S^2 T^{2\alpha} \left[ \frac{1}{\alpha^2} (\mu_1 l + \mu_2 B_1^*) + \frac{1}{T(2\alpha - 1)} (\nu_1 l + \nu_2 B_2^*) \right] \right) < 1,
\]

where \( B_1^* = \sup_{t \in [0, t_1]} \int_0^t K(t, s) ds < \infty \) and \( B_2^* = \sup_{t \in [0, t_1]} \int_0^t P(t, s) ds < \infty \).
Proof. Define the operator $\mathcal{P} : \mathcal{B}_0 \to \mathcal{B}_0$ by

$$
(\mathcal{P}x)(t) = \begin{cases}
\phi(t), & t \in (-\infty, 0], \\
\int_0^t S_\alpha(t-s)f(s, x_s, B_1x(s))ds & + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2x(s))d\nu(s), & t \in [0, t_1], \\
T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)f(s, x_s, B_1x(s))ds & + \int_{t_1}^t S_\alpha(t-s)\sigma(s, x_s, B_2x(s))d\nu(s), & t \in (t_1, t_2], \\
\vdots & \vdots & \\
T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)f(s, x_s, B_1x(s))ds & + \int_{t_m}^t S_\alpha(t-s)\sigma(s, x_s, B_2x(s))d\nu(s), & t \in (t_m, T].
\end{cases}
$$

For $\phi \in \mathcal{B}_0$, define

$$
g(t) = \begin{cases}
\phi(t), & t \in (-\infty, 0], \\
0, & t \in J.
\end{cases}
$$

then $g_0 = \phi$. Next we define the function

$$
\overline{z}(t) = \begin{cases}
0, & t \in (-\infty, 0], \\
z(t), & t \in J.
\end{cases}
$$

for each $z \in C(J, \mathbb{R})$ with $z(0) = 0$. If $x(\cdot)$ satisfies (3.1) then $x(t) = g(t) + \overline{z}(t)$ for $t \in J$, which implies $x_t = g_t + \overline{z}_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$
z(t) = \begin{cases}
\int_0^t S_\alpha(t-s)f(s, g_s + \overline{z}_s, B_1(g(s) + \overline{z}(s)))ds & + \int_0^t S_\alpha(t-s)\sigma(s, g_s + \overline{z}_s, B_2(g(s) + \overline{z}(s)))d\nu(s), & t \in [0, t_1], \\
T_\alpha(t-t_1)((g(t_1^-) + \overline{z}(t_1^-)) + I_1(g(t_1^-) + \overline{z}(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)f(s, g_s + \overline{z}_s, B_1(g(s) + \overline{z}(s)))ds & + \int_{t_1}^t S_\alpha(t-s)\sigma(s, g_s + \overline{z}_s, B_2(g(s) + \overline{z}(s)))d\nu(s), & t \in (t_1, t_2], \\
\vdots & \vdots & \\
T_\alpha(t-t_m)((g(t_m^-) + \overline{z}(t_m^-)) + I_m(g(t_m^-) + \overline{z}(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)f(s, g_s + \overline{z}_s, B_1(g(s) + \overline{z}(s)))ds & + \int_{t_m}^t S_\alpha(t-s)\sigma(s, g_s + \overline{z}_s, B_2(g(s) + \overline{z}(s)))d\nu(s), & t \in (t_m, T].
\end{cases}
$$

Set $\mathcal{B}_0^0 = \{z \in \mathcal{B}_0, \text{ such that } z_0 = 0\}$ and for any $z \in \mathcal{B}_0^0$, we have

$$
\|z\|_{\mathcal{B}_0} = \|z_0\|_{\mathcal{B}_0} + \sup_{t \in J} E\|z(t)\|^2)^{1/2} = \sup_{t \in J} E\|z(t)\|^2)^{1/2},
$$

thus $(\mathcal{B}_0^0, \| \cdot \|_{\mathcal{B}_0})$ is a Banach space.
Define the operator $\Psi : \mathcal{B}_0 \rightarrow \mathcal{B}_0$ by

\[
(\Psi z)(t) = \begin{cases} 
    \int_0^t S_\alpha(t-s)f(s, g_i + \bar{z}, B_1(g(s) + \bar{z}(s)))\, ds \\
    + \int_0^t S_\alpha(t-s)\sigma(s, g_i + \bar{z}, B_2(g(s) + \bar{z}(s)))\, dw(s), & t \in [0, t_1], \\
    T_\alpha(t-t_1)(z(t_1^-) + I_1(z(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)f(s, g_i + \bar{z}, B_1(g(s) + \bar{z}(s)))\, ds \\
    + \int_{t_1}^t S_\alpha(t-s)\sigma(s, g_i + \bar{z}, B_2(g(s) + \bar{z}(s)))\, dw(s), & t \in (t_1, t_2], \\
    \vdots \\
    T_\alpha(t-t_m)(z(t_m^-) + I_m(z(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)f(s, g_i + \bar{z}, B_1(g(s) + \bar{z}(s)))\, ds \\
    + \int_{t_m}^t S_\alpha(t-s)\sigma(s, g_i + \bar{z}, B_2(g(s) + \bar{z}(s)))\, dw(s), & t \in (t_m, T].
\end{cases}
\]

(3.5)

In order to prove the existence result, it is enough to show that $\Psi$ has a unique fixed point. Let $z, z^* \in \mathcal{B}_0$, then for all $t \in [0, t_1]$, we have

\[
E \left\| (\Psi z)(t) - (\Psi z^*)(t) \right\|^2 \leq 2E \left\| \int_0^t S_\alpha(t-s)\left[ f(s, g_i + \bar{z}, B_1(g(s) + \bar{z}(s))) - f(s, g_i + \bar{z^*}, B_1(g(s) + \bar{z^*}(s))) \right] \, ds \right\|^2 \\
+ 2E \left\| \int_0^t S_\alpha(t-s)\left[ \sigma(s, g_i + \bar{z}, B_2(g(s) + \bar{z}(s))) - \sigma(s, g_i + \bar{z^*}, B_2(g(s) + \bar{z^*}(s))) \right] \, dw(s) \right\|^2 \\
\leq 2 \int_0^t \left\| S_\alpha(t-s) \right\|_2 \int_0^t \left\| S_\alpha(t-s) \right\|_2 E \left\| f(s, g_i + \bar{z}, B_1(g(s) + \bar{z}(s))) - f(s, g_i + \bar{z^*}, B_1(g(s) + \bar{z^*}(s))) \right\|^2 \, ds \\
+ 2 \int_0^t \left\| S_\alpha(t-s) \right\|^2_2 \left\| \sigma(s, g_i + \bar{z}, B_2(g(s) + \bar{z}(s))) - \sigma(s, g_i + \bar{z^*}, B_2(g(s) + \bar{z^*}(s))) \right\|^2 \, ds \\
\leq 2 \tilde{M}_2^2 \int_0^t (t-s)^{\alpha-1} ds \int_0^t (t-s)^{\alpha-1} \left\| \mu_1 \| \bar{z} - \bar{z^*} \|_{\mathcal{B}_0}^2 \\
+ \mu_2 E \left\| B_1(g(s) + \bar{z}(s)) - B_1(g(s) + \bar{z^*}(s)) \right\|_2^2 \, ds \\
+ 2 \tilde{M}_2^2 \int_0^t (t-s)^{2(\alpha-1)} \left[ \| \bar{z} - \bar{z^*} \|_{\mathcal{B}_0}^2 + v_2 E \left\| B_2(g(s) + \bar{z}(s)) - B_2(g(s) + \bar{z^*}(s)) \right\|_2^2 \right] \, ds \\
\leq 2 \tilde{M}_2^2 T_\alpha \int_0^t (t-s)^{\alpha-1} \left\| \mu_1 I \sup E \| z(s) - z^*(s) \|_{\mathcal{B}_0}^2 \\
+ \mu_2 B_1^* \sup E \| z(s) - z^*(s) \|_{H^1_0}^2 \right\|_2^2 \, ds \\
+ 2 \tilde{M}_2^2 \int_0^t (t-s)^{2(\alpha-1)} \left[ \| \bar{z} - \bar{z^*} \|_{\mathcal{B}_0}^2 + v_2 B_2^* \sup E \| z(s) - z^*(s) \|_{H^1_0}^2 \right] \, ds \\
\leq 2 \tilde{M}_2^2 T_\alpha \left\| \mu_1 I + \mu_2 B_1^* \right\|_2 \| z(s) - z^*(s) \|^2_{\mathcal{B}_0} + 2 \tilde{M}_2^2 T_\alpha^{\alpha-1} \left[ \| \bar{z} - \bar{z^*} \|^2_{\mathcal{B}_0} + v_2 \right] \| \bar{z} - \bar{z^*} \|^2_{\mathcal{B}_0} \]

Thus, for all \( t \), we have

\[
\|e(t)\|_2 \leq 4\|e(t_0)\|_2 + 4\int_{t_0}^t \|e(s)\|_2 ds + 4 \int_{t_0}^t \|e'\|_2 ds + 4 \int_{t_0}^t \|e''\|_2 ds + 4 \int_{t_0}^t \|e'''\|_2 ds.
\]

By the Gronwall–Bellman lemma, we get

\[
\|e(t)\|_2 \leq \frac{4\|e(t_0)\|_2}{1 - \frac{3}{2}h}.
\]

Similarly, when \( t \in [t_i, t_{i+1}] \), we get

\[
\|e(t)\|_2 \leq \frac{4\|e(t_i)\|_2}{1 - \frac{3}{2}h}.
\]

Thus, for all \( t \), we have

\[
\|e(t)\|_2 \leq \frac{4\|e(t_0)\|_2}{1 - \frac{3}{2}h}.
\]
The second result is established using Krasnoselskii’s fixed point theorem. Now, we make the following assumptions.

(H5) $f : J \times B_{\mathcal{H}} \times \mathbb{H} \to \mathbb{H}$ is continuous and there exist two continuous functions $\mu_1, \mu_2 : J \to (0, \infty)$ such that
\[
E\|f(t, \psi, x)\|_{\mathbb{H}}^2 \leq \mu_1(t)\|\psi\|_{B_{\mathcal{H}}}^2 + \mu_2(t)E\|x\|_{\mathbb{H}}^2, \quad (t, \psi, x) \in J \times B_{\mathcal{H}} \times \mathbb{H},
\]
where $\mu_1^* = \sup_{t \in [0,1]} \mu_1(s)$ and $\mu_2^* = \sup_{t \in [0,1]} \mu_2(s)$.

(H6) $\sigma : J \times B_{\mathcal{H}} \times \mathcal{L}^0_2 \to \mathbb{H}$ is continuous and there exist two continuous functions $v_1, v_2 : J \to (0, \infty)$ such that
\[
E\|\sigma(t, \psi, x)\|_{\mathbb{H}}^2 \leq v_1(t)\|\psi\|_{B_{\mathcal{H}}}^2 + v_2(t)E\|x\|_{\mathbb{H}}^2, \quad (t, \psi, x) \in J \times B_{\mathcal{H}} \times \mathcal{L}^0_2,
\]
where $v_1^* = \sup_{t \in [0,1]} v_1(s)$ and $v_2^* = \sup_{t \in [0,1]} v_2(s)$.

(H7) The function $l_k : \mathbb{H} \to \mathbb{H}$ is continuous and there exists $\Lambda > 0$ such that
\[
\Lambda = \max_{1 \leq k \leq m, x \in B_q} \{E\|l_k(x)\|_{C_0}^2\},
\]
where $B_q = \{y \in B_0^q, \|y\|_{C_0}^2 \leq q, q > 0\}$.

The set $B_q$ is clearly a bounded closed convex set in $\mathbb{L}^0_0$ for each $q$ and for each $y \in B_q$. From Lemma 2.1, we have
\[
\|y_t + z_t\|_{\mathbb{H}}^2 \leq 2\left(\|y_0\|_{C_0}^2 + \|z_0\|_{\mathbb{H}}^2\right) \\
\leq 4\left(\bar{p}^2 \sup_{t \in [0,1]} E\|y(t)\|_{\mathbb{H}}^2 + \|y_0\|_{\mathbb{H}}^2\right) + 4\left(\bar{p}^2 \sup_{t \in [0,1]} E\|y(t)\|_{\mathbb{H}}^2 + \|z_0\|_{\mathbb{H}}^2\right) \\
\leq 4\left(\|\phi\|_{\mathbb{H}}^2 + \bar{p}^2 q\right).
\]

**Theorem 3.3.** Suppose that the assumptions (H1)–(H3) and (H5)–(H7) are satisfied with
\[
q \geq \frac{\tilde{M}_1^2}{\bar{p}^2}(q + \Lambda) + \frac{\tilde{M}_2^2T^{2\alpha}}{\lambda_2^2 + \frac{\lambda_1}{T(2\alpha - 1)}} \left[\frac{1}{\alpha^2} + \frac{1}{T(2\alpha - 1)}(v_1 I_1 + v_2 B_2^*)\right] < 1.
\]

Then the impulsive stochastic fractional differential equation (2.1) has at least one mild solution on $(-\infty, T]$.

**Proof.** Let $\Theta_1 : B_q \to B_q$ and $\Theta_2 : B_q \to B_q$ be defined as
\[
(\Theta_1 z)(t) = \begin{cases} 
0, & t \in [0, t_1], \\
I_{t_1}^t(\psi, B_1(g(s) + \bar{z})), & t \in (t_1, t_2], \\
\vdots, & \\
I_{t_m}^t(\psi, B_1(g(s) + \bar{z})), & t \in (t_m, T].
\end{cases}
\]
and
\[
(\Theta_2 z)(t) = \begin{cases} 
\int_0^t S_{\alpha}(t - s)\sigma(s, g_s + \bar{z}, B_1(g(s) + \bar{z}))ds \\
+ \int_0^{t_1} S_{\alpha}(t - s)\sigma(s, g_s + \bar{z}, B_2(g(s) + \bar{z})))dw(s), & t \in [0, t_1], \\
\vdots, & \\
\int_{t_1}^t S_{\alpha}(t - s)\sigma(s, g_s + \bar{z}, B_2(g(s) + \bar{z})))dw(s), & t \in (t_1, t_2], \\
\vdots, & \\
\int_{t_m}^t S_{\alpha}(t - s)\sigma(s, g_s + \bar{z}, B_1(g(s) + \bar{z})))ds \\
+ \int_{t_m}^t S_{\alpha}(t - s)\sigma(s, g_s + \bar{z}, B_2(g(s) + \bar{z})))dw(s), & t \in (t_m, T].
\end{cases}
\]

In order to use Theorem 2.7 we will verify that $\Theta_1$ is compact and continuous while $\Theta_2$ is a contraction operator. For the sake of convenience, we divide the proof into several steps.
Step 1. We show that $\Theta_1 z + \Theta_2 z^* \in B_q$ for $z, z^* \in B_q$. For $t \in [0, t_1]$, we have

\[
E \left\| (\Theta_1 z)(t) + (\Theta_2 z^*)(t) \right\|_{\mathbb{H}}^2 \leq 2E \left\| \int_0^t S_u(t - s)f(s, g_s + Z^*_s, B_1(g(s) + Z^*(s)))ds \right\|_{\mathbb{H}}^2
+ 2E \left\| \int_0^t S_u(t - s)\sigma(s, g_s + Z^*_s, B_2(g(s) + Z^*(s)))dw(s) \right\|_{\mathbb{H}}^2
\leq 2 \int_0^t \|S_u(t - s)\|ds \int_0^t \|S_u(t - s)\|E \left\| f(s, g_s + Z^*_s, B_1(g(s) + Z^*(s))) \right\|_{\mathbb{H}}^2 ds
+ 2 \int_0^t \|S_u(t - s)\|2E \left\| \sigma(s, g_s + Z^*_s, B_2(g(s) + Z^*(s))) \right\|_{\mathbb{H}}^2 ds
\leq 2\tilde{M}_s^2 \int_0^t (t - s)^{\alpha - 1}ds \int_0^t (t - s)^{\alpha - 1}\|\mu_1(s)\|g_s + Z^*_s\|_{\mathbb{H}}^2
+ \mu_2(s)E \|B_1(g(s) + Z^*(s))\|_{\mathbb{H}}^2 ds
+ 2\tilde{M}_s^2 \int_0^t (t - s)^{2(\alpha - 1)}[\mu_1(s)\|g_s + Z^*_s\|_{\mathbb{H}}^2 + \nabla_v \alpha \|B_2(g(s) + Z^*(s))\|_{\mathbb{H}}^2] ds
\leq 2\tilde{M}_s^2 \int_0^t (t - s)^{\alpha - 1}[4\mu_1^2(\|\phi\|_{\mathbb{H}}^2 + \|\phi\|_{\mathbb{H}}^2 q) + \mu_2^2(\|B_1(g(s) + Z^*(s))\|_{\mathbb{H}}^2) ds
+ 2\tilde{M}_s^2 \int_0^t (t - s)^{2(\alpha - 1)}[4
\n\nThus, by the condition (3.7), we obtain $\|\Theta_1 z + \Theta_2 z^*\|_{g^0} \leq q$.

Similarly, for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have the estimate

\[
E \left\| (\Theta_1 z)(t) + (\Theta_2 z^*)(t) \right\|_{\mathbb{H}}^2 \leq 4\|T_u(t - t_i)\|^2E \left\| z(t_i^-) \right\|_{\mathbb{H}}^2 + 4\|T_u(t - t_i)\|^2E \left\| I_i(z(t_i^-)) \right\|_{\mathbb{H}}^2
+ 4E \left\| \int_{t_i}^t S_u(t - s)f(s, g_s + Z^*_s, B_1(g(s) + Z^*(s)))ds \right\|_{\mathbb{H}}^2
+ 4E \left\| \int_{t_i}^t S_u(t - s)\sigma(s, g_s + Z^*_s, B_2(g(s) + Z^*(s)))dw(s) \right\|_{\mathbb{H}}^2
\leq 4\tilde{M}_s^2 \left\| z \right\|_{\mathbb{H}}^2 q + 4\tilde{M}_s^2 \left\| I_i(z(t_i^-)) \right\|_{\mathbb{H}}^2
+ 4E \left\| \int_{t_i}^t S_u(t - s)f(s, g_s + Z^*_s, B_1(g(s) + Z^*(s)))ds \right\|_{\mathbb{H}}^2
+ 4E \left\| \int_{t_i}^t S_u(t - s)\sigma(s, g_s + Z^*_s, B_2(g(s) + Z^*(s)))dw(s) \right\|_{\mathbb{H}}^2
\leq 4\tilde{M}_s^2 (q + \Lambda) + 4\tilde{M}_s^2 + \frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)}
\leq q.
\]

This implies that $\|\Theta_1 z + \Theta_2 z^*\|_{g^0} \leq q$ with $\lambda_1 = 4\mu_1^2(\|\phi\|_{\mathbb{H}}^2 + \|\phi\|_{\mathbb{H}}^2 q) + \mu_2^2(\|B_1(g(s) + Z^*(s))\|_{\mathbb{H}}^2$ and $\lambda_2 = 4\nabla_v \alpha \|B_2(g(s) + Z^*(s))\|_{\mathbb{H}}^2 + \nabla_v \alpha \|B_2(g(s) + Z^*(s))\|_{\mathbb{H}}^2$. Hence, we get $\Theta_1 z + \Theta_2 z^* \in B_q$.

Step 2. The map $\Theta_1$ is continuous on $B_q$.

Let $(z^n)_{n=1}^\infty$ be a sequence in $B_q$ with $\lim z^n \rightarrow z \in B_q$. Then for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, we have

\[
E \left\| (\Theta_1 z^n)(t) - (\Theta_1 z)(t) \right\|_{\mathbb{H}}^2 \leq 2\|T_u(t - t_i)\|^2 \left\| z^n(t_i^-) - z(t_i^-) \right\|_{\mathbb{H}}^2
+ 2\|T_u(t - t_i)\|^2 \left\| I_i(z^n(t_i^-)) - I_i(z(t_i^-)) \right\|_{\mathbb{H}}^2.
\]

Since the functions $I_i$, $i = 1, 2, \ldots, m$ are continuous, hence $\lim_{n \rightarrow \infty} E \|\Theta_1 z^n - \Theta_1 z\|_{\mathbb{H}}^2 = 0$ which implies that the mapping $\Theta_1$ is continuous on $B_q$. 

Step 3. $\Theta_1$ maps bounded sets into bounded sets in $B_q$.

Let us prove that for $q > 0$ there exists a $\hat{r} > 0$ such that for each $z \in B_q$, we have $E \| (\Theta_1 z)(t) \|_{H}^2 \leq \hat{r}$ for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$. Now, we have

$$E \| (\Theta_1 z)(t) \|_{H}^2 \leq 2 \| T_a(t - t_i) \| \left[ E \| z(t_i^-) \|_{H}^2 + E \| I_z(t(t_i^-)) \|_{H}^2 \right]$$

$$\leq 2 \hat{M}_2^2 (q + A) = \hat{r},$$

which proves the desired result.

Step 4. The map $\Theta_1$ is equicontinuous.

Let $u, v \in (t_i, t_{i+1}]$, $t_i \leq u < v \leq t_{i+1}$, $i = 0, 1, \ldots, m, z \in B_q$, we obtain

$$E \| (\Theta_1 z)(v) - (\Theta_1 z)(u) \|_{H}^2 \leq 2 \| T_a(v - t_i) - T_a(u - t_i) \| \left[ E \| z(t_i^-) \|_{H}^2 + E \| I_z(t(t_i^-)) \|_{H}^2 \right]$$

$$\leq 2(q + A) \| T_a(v - t_i) - T_a(u - t_i) \|^2.$$

Since $T_a$ is strongly continuous and it allows us to conclude that $\lim_{u \to v} \| T_a(v - t_i) - T_a(u - t_i) \|^2 = 0$, which implies that $\Theta_1(B_q)$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli’s theorem, we conclude that the operator $\Theta_1$ is compact.

Now, it only remains to show that the map $\Theta_2$ is a contraction mapping. Let $z, z^* \in B_q$ and $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$E \| (\Theta_2 z)(t) - (\Theta_2 z^*)(t) \|_{H}^2 \leq 2E \left\| \int_{t_i}^t S_a(t - s) \left[ f(s, g_s + Z_s, B_1(g(s) + Z(s))) - f(s, g_s + Z_s, B_1(g(s) + Z^*(s))) \right] ds \right\|_{H}^2$$

$$+ 2E \left\| \int_{t_i}^t S_a(t - s) \left[ \sigma(s, g_s + Z_s, B_2(g(s) + Z(s))) - \sigma(s, g_s + Z_s, B_2(g(s) + Z^*(s))) \right] dw(s) \right\|_{H}^2$$

$$\leq 2 \left\| \int_{t_i}^t S_a(t - s) \|f(s, g_s + Z_s, B_1(g(s) + Z(s))) - f(s, g_s + Z_s, B_1(g(s) + Z^*(s)))\|_{H}^2 \| ds \right.$$

$$+ 2 \left\| \int_{t_i}^t S_a(t - s) \|\sigma(s, g_s + Z_s, B_2(g(s) + Z(s))) - \sigma(s, g_s + Z_s, B_2(g(s) + Z^*(s)))\|_{H}^2 \| ds \right.$$
By the condition (3.8), we obtain that \( \Theta_2 \) is a contraction mapping. Hence, by Krasnoselskii’s fixed point theorem we can conclude that the problem (2.1) has at least one solution on \((−∞, T]\). This completes the proof of the theorem. \( \square \)

**Example 3.4.** In this section, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

\[
D^q_t u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \int_{-\infty}^{t} H(t, x, s - t)Q(u(s, x))ds + \int_0^{t} k(s, t)e^{-u(s,x)}ds \\
+ \left[ \int_{-\infty}^{t} V(t, x, s - t)U(u(s, x))ds + \int_0^{t} p(s, t)e^{-u(s,x)}ds \right] \frac{d\beta(t)}{dt},
\]

where \( \beta(t) \) is a standard cylindrical Wiener process in \( \mathbb{H} \) defined on a stochastic space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}) \); \( D^q_t \) is the Caputo fractional derivative of order \( 0 < q < 1 \); \( 0 < t_1 < t_2 < \cdots < t_n < b \) are prefixed numbers; \( H, Q, V \) and \( U \) are continuous; \( \phi \in \mathcal{B}_h \).

Let \( \mathbb{H} = L^2([0, \pi]) \) with the norm \( \| \cdot \| \). Define \( A : \mathbb{H} \to \mathbb{H} \) by \( Az = z'' \) with the domain \( D(A) = \{ z \in \mathbb{H}, z, z' \text{ are absolutely continuous}, z'' \in \mathbb{H}, \text{ and } z(0) = z'(\pi) = 0 \} \). Then

\[
Az = \sum_{n=1}^{\infty} n^2 (z, \zeta_n)\zeta_n, \quad z \in D(A),
\]

where \( \zeta_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \ n \in \mathbb{N} \) is the orthogonal set of eigenvectors of \( A \). It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( (T(t))_{t \geq 0} \) in \( \mathbb{H} \) and is given by

\[
T(t)z = \sum_{n=1}^{\infty} e^{-n^2t}(z, \zeta_n)\zeta_n, \quad \text{for all } z \in \mathbb{H}, \ t > 0.
\]

It follows from the above expressions that \( (T(t))_{t \geq 0} \) is a uniformly bounded compact semigroup, so that, \( R(\lambda, A) = (\lambda - A)^{-1} \) is a compact operator for all \( \lambda \in \rho(A) \) i.e. \( A \in \mathcal{A}^\omega (\theta_0, \omega_0) \). Let \( h(s) = e^{2s}, \ s < 0 \), then \( l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} \) and define

\[
\| \phi \|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E |\phi(\theta)|^2)^{1/2}ds.
\]

Hence for \((t, \phi) \in [0, b] \times \mathcal{B}_h \), where \( \phi(\theta)(z) = \phi(\theta, z), (\theta, z) \in (−\infty, 0] \times [0, \pi] \). Put \( u(t) = u(t, \cdot) \), that is \( u(t)(x) = u(t, x) \). Define \( f : \mathcal{B}_h \times L^2([0, \pi]) \to L^2([0, \pi]) \) and \( \sigma : \mathcal{B}_h \times L^2([0, \pi]) \to L^2([0, \pi]) \) as follows:

\[
f(t, \phi, B_1 u(t)) = \int_{-\infty}^{0} H(t, x, \theta)Q(\phi(\theta))(x)d\theta + B_1 u(t)(x),
\]

\[
\sigma(t, \phi, B_2 u(t)) = \int_{-\infty}^{0} V(t, x, \theta)U(\phi(\theta))(x)d\theta + B_2 u(t)(x),
\]

where \( B_1 u(t)(x) = \int_{0}^{t} k(s, t)e^{-u(s,x)}ds \) and \( B_2 u(t)(x) = \int_{0}^{t} p(s, t)e^{-u(s,x)}ds \). Then, with the above settings the considered equation (3.11) can be written in the abstract form of Eq. (2.1). All conditions of Theorem 3.2 are now fulfilled, so we deduce that the system (3.11) has a mild solution on \((−\infty, T]\).

### 4. Stochastic fractional equations with nonlocal conditions

The initial conditions usually represent the measurements at initial time. However, in various real world problems, it is possible to require more measurements at some instances in addition to standard initial data and, therefore, the initial conditions changed to nonlocal conditions. Balasubramaniam et al. [36] studied the existence of solutions for semilinear
neutral stochastic functional differential equations with nonlocal conditions. In the last few years, there has been an increasing interest in study of fractional differential equations involving nonlocal conditions [34]. In this section, we are concerned with the existence and uniqueness of mild solutions for semilinear stochastic fractional differential equations with nonlocal conditions in the form

$$D^\alpha_t x(t) + Ax(t) = f(t, x(t)) + \sigma(t, x(t)) \frac{dw(t)}{dt}, \quad t \in J = [0, T], \quad 0 < \alpha < 1,$$

$$x(0) + g(x) = x_0,$$

where $A; D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$; $D^\alpha_t$ and $w$ are defined as in Eq. (2.1). Further, the functions $f$, $\sigma$ and $g$ are given functions to be defined later. The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L^2(\Omega, \mathbb{H})$, is a Banach space equipped with norm $\|x(\cdot)\|_{L^2} = (E \parallel x(\cdot, w)\|_\mathbb{H}^2)^{1/2}$, where $E$ is defined by $E(\cdot) = \int_\Omega h(\omega)dP$, $w \in \Omega$. Let $C(J, L^2(\Omega, \mathbb{H}))$ be the Banach space of all continuous maps from $J$ into $L^2(\Omega, \mathbb{H})$ satisfying the condition $\sup_{t \in J} E\|x(t)\|^2 < \infty$.

Let $H_2$ be the closed subspace of all continuous processes $x$ that belong to the space $C(J, L^2(\Omega, \mathbb{H}))$ consisting of $F_\tau$-adapted measurable processes such that the $F_0$-adapted processes $x(0)$ with a seminorm $\| \cdot \|_{H_2}$ in $H_2$ be defined by

$$\|x\|_{H_2} = \left( \sup_{t \in J} \|x(t)\|_{L^2}^2 \right)^{1/2}.$$

It is easy to verify that $H_2$ furnished with the norm topology as defined above, is a Banach space.

Now, we define the definition of the mild solution of (4.1).

**Definition 4.1.** A continuous stochastic process $x : J \rightarrow \mathbb{H}$ is called a mild solution of (4.1) if the following conditions hold.

(i) $x(t)$ is measurable and $F_\tau$-adapted.

(ii) $x(0) + g(x) = x_0$.

(iii) $x$ satisfies the following equation

$$x(t) = T_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)f(s, x(s))ds + \int_0^t S_\alpha(t-s)\sigma(s, x(s))dw(s),$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\gamma_t} e^{z(t-s)}\frac{dz}{z^{1-\alpha}}$, $S_\alpha(t) = t^{a-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\gamma_t} e^{z(t-s)}\frac{dz}{z^{1-\alpha}}$, $\gamma_t$ denotes the Bromwich path, $S_\alpha(t)$ is called the $\alpha$-resolvent family and $T_\alpha(t)$ is the solution operator generated by $-A$.

Further, we assume the following conditions.

(H8) There exists a constant $L_y > 0$ such that $E\|g(x) - g(y)\|_{\mathbb{H}}^2 \leq L_y \|x - y\|_{\mathbb{H}}^2$.

(H9) The nonlinear maps $f : J \times \mathbb{H} \rightarrow \mathbb{H}$ and $\sigma : J \times \mathbb{H} \rightarrow L^0_2$ are continuous and there exist constants $L_f$ and $L_\sigma$ such that

$$E\|f(t, x) - f(t, y)\|_{\mathbb{H}}^2 \leq L_f \|x - y\|_{\mathbb{H}}^2,$$

$$E\|\sigma(t, x) - \sigma(t, y)\|_{L^0_2}^2 \leq L_\sigma \|x - y\|_{\mathbb{H}}^2$$

for all $x, y \in \mathbb{H}$ and $t \in [0, T]$.

(H10) $f \in C(J \times \mathbb{H}, \mathbb{H})$, $g \in C(\mathbb{H}, \mathbb{H})$, and $\sigma \in C(J \times \mathbb{H}, L^0_2)$. Moreover, there exists a constant $C_1 > 0$ such that for $x \in \mathbb{H}$,

$$E\|g(x)\|_{\mathbb{H}}^2 \leq C_1$$

and for $s \in J$, $x \in B$, there exist two continuous functions $\tilde{L}_f$, $\tilde{L}_\sigma : J \rightarrow (0, \infty)$ such that

$$E\|f(t, x)\|_{\mathbb{H}}^2 \leq \tilde{L}_f(t)\phi(E\|x\|_{\mathbb{H}}^2), \quad E\|\sigma(t, x)\|_{L^0_2}^2 \leq \tilde{L}_\sigma(t)\psi(E\|x\|_{\mathbb{H}}^2).$$

(H11)

$$\int_0^T \xi(s)ds \leq \int_0^\infty \frac{ds}{\phi(s) + \psi(s)},$$

where

$$\xi(t) = \max \left\{ \frac{4\tilde{M}_t^2 T^\alpha}{\alpha} t^{\alpha-1}\tilde{L}_f(t), \frac{4\tilde{M}_t^2}{\alpha} t^{2(\alpha-1)}\tilde{L}_\sigma(t) \right\}, \quad c = 4\tilde{M}_t^2 (E\|x_0\|_{\mathbb{H}}^2 + C_1).$$

**Theorem 4.2.** Under the assumptions (H1), (H8) and (H9), the fractional stochastic equation (4.1) has a unique mild solution on $J$ provided that

$$3\tilde{M}_t^2 L_y + 3\tilde{M}_t^2 T^{2\alpha} \left( \frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha - 1)} \right) \leq 1.$$  

(4.2)
Proof. Let \( \lambda : H_2 \rightarrow H_2 \) be the operator defined by

\[
(\lambda x)(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s) f(s, x(s)) \, ds + \int_0^t S_\alpha(t-s) \sigma(s, x(s)) \, dw(s).
\]

It can be seen that \( \lambda \) maps \( H_2 \) into itself. Let us show that \( \lambda \) is a contraction on \( H_2 \). For \( t \in J \), it follows from the assumptions (H1), (H8) and (H9) that

\[
E \| \lambda(x(t) - \lambda y(t)) \|_{H}^2 \leq 3 \| T_\alpha(t) \|^2 E \| g(x) - g(y) \|_{H}^2 + 3 \int_0^t \| S_\alpha(t-s) \| \| f(s, x(s)) - f(s, y(s)) \|_{H}^2 \, ds \\
+ 3 \int_0^t \| S_\alpha(t-s) \|^2 \| \sigma(s, x(s)) - \sigma(s, y(s)) \|_{L_2}^2 \, ds
\]

\[
\leq 3 \tilde{M}_2^2 t \| x - y \|_{H}^2 + 3 \tilde{M}_2^2 \frac{T}{\alpha^2} \int_0^t (t-s)^{\alpha-1} L_t E \| x - y \|_{H}^2 \, ds
\]

Hence by the condition (4.2), \( \lambda \) is a contraction mapping. Therefore, by the Banach contraction principle \( \lambda \) has a unique fixed point. The proof is complete. \( \square \)

Next our second result is based on the following Schaefer’s fixed point theorem.

Theorem 4.3. Let \( \hat{K} \) be a closed convex subset of a Banach space \( X \) such that \( 0 \in \hat{K} \). Let \( P : \hat{K} \rightarrow \hat{K} \) be a completely continuous map. Then the set \( \{ x \in \hat{K} : x = \nu P x; 0 \leq \nu \leq 1 \} \) is unbounded or \( P \) has a fixed point.

Theorem 4.4. Assume that (H1), (H8), (H10) and (H11) hold. Then the fractional stochastic equation (4.1) has at least one mild solution on \( [0, T] \).

Proof. Define the operator \( \lambda : H_2 \rightarrow H_2 \) as in Theorem 4.2. Now, we have to prove that \( \lambda \) is a completely continuous operator. Note that \( \lambda \) is well defined in \( H_2 \). For the sake of convenience, we divide the proof into several steps.

Step 1. We prove that \( \lambda \) is continuous.

Let \( \{ x^n \}_{n=0}^{\infty} \) be a sequence in \( H_2 \) such that \( x^n \rightarrow x \) in \( H_2 \). Since the functions \( f, g \) and \( \sigma \) are continuous,

\[
\lim_{n \rightarrow \infty} E \| \lambda x^n(t) - \lambda x(t) \|_{H}^2 = 0
\]

in \( H_2 \) for every \( t \in J \). This implies that the mapping \( \lambda \) is continuous on \( H_2 \).

Step 2. Next we prove that \( \lambda \) maps bounded sets into bounded sets in \( H_2 \).

To prove that for any \( r > 0 \), there exists a \( \gamma > 0 \) such that for \( x \in B_r = \{ x \in H_2 : E \| x \|_{H}^2 \leq r \} \), we have \( E \| \lambda x \|_{H}^2 \leq \gamma \). For any \( x \in B_r, \ t \in J \), we have

\[
E \| \lambda x(t) \|_{H}^2 \leq 4 \| T_\alpha(t) \|^2 E \| x_0 \|_{H}^2 + 4 \| T_\alpha(t) \|^2 E \| g(x) \|_{H}^2 + 4 \int_0^t \| S_\alpha(t-s) \| \| f(s, x(s)) \|_{H}^2 \, ds \\
+ 4 \int_0^t \| S_\alpha(t-s) \|^2 \| \sigma(s, x(s)) \|_{L_2}^2 \, ds
\]

\[
\leq 4 \tilde{M}_2^2 r + 4 \tilde{M}_2^2 C_1 + 4 \tilde{M}_2^2 \frac{T}{\alpha^2} \int_0^t (t-s)^{\alpha-1} L_t \, ds
\]

\[
+ 4 \tilde{M}_2 \psi(r) \int_0^t (t-s)^{\alpha-1} L_t \, ds
\]

\[
= \gamma, \quad t \in J.
\]
Step 3. We show that \( \lambda \) maps bounded sets into equicontinuous sets of \( B_r \).
Let \( 0 \leq u < v \leq T \), for each \( x \in B_r \), we have
\[
E\|\lambda x(v) - \lambda x(u)\|^2 \leq 6\|T_a(v) - T_a(u)\|^2 \|x_0\|^2 + 6\|T_a(v) - T_a(u)\|^2 E\|g(x)\|^2 \\
+ 6E \int_0^v [S_a(v - s) - S_a(u - s)]f(s, x(s))ds \|H_a\|^2 \\
+ 6E \int_0^v S_a(v - s)f(s, x(s))ds \|H_a\|^2 \\
+ 6E \int_0^v [S_a(v - s) - S_a(u - s)]\sigma(s, x(s))dw(s) \|H_a\|^2 \\
+ 6E \int_0^v S_a(v - s)\sigma(s, x(s))dw(s) \|H_a\|^2.
\]
Therefore we obtain
\[
E\|\lambda x(v) - \lambda x(u)\|^2 \leq 6(r + C_1)\|T_a(v) - T_a(u)\|^2 \\
+ 6 \int_0^u \|S_a(v - s) - S_a(u - s)\|ds \\
\times \int_0^v \|S_a(v - s) - S_a(u - s)\|E\|f(s, x(s))\|^2 \|H_a\|^2 ds \\
+ 6 \int_0^u \|S_a(v - s)\|ds \int_0^v \|S_a(v - s)\|E\|f(s, x(s))\|^2 \|H_a\|^2 ds \\
+ 6 \int_0^u \|S_a(v - s) - S_a(u - s)\|^2 E\|\sigma(s, x(s))\| \|H_a\|^2 ds \\
+ \int_0^v \|S_a(v - s)\|^2 E\|\sigma(s, x(s))\| \|H_a\|^2 ds.
\]
Since \( T_a(t) \) and \( S_a(t) \) are strongly continuous, \( \|T_a(v) - T_a(u)\| \to 0 \) and \( \|S_a(v - s) - S_a(u - s)\| \to 0 \) as \( u \to v \). Thus, from the above inequality we have \( \lim_{u \to v} E\|\lambda x(v) - \lambda x(u)\|^2 = 0 \). Thus, the set \( \{\lambda x, x \in B_r\} \) is equicontinuous. Finally, combining Step 1 to Step 3 with Ascoli’s theorem, we conclude that the operator \( \lambda \) is compact.

Step 4. Next, we show that the set
\[ N = \{x \in H_2 \text{ such that } x = q\lambda x(t) \text{ for some } 0 < q < 1\} \]
is bounded. Let \( x \in N \) then \( x(t) = q\lambda x(t) \) for some \( 0 < q < 1 \). Then for each \( t \in J \), we have
\[
x(t) = q \left( T_a(t)(x_0 - g(x)) + \int_0^t S_a(t - s)f(s, x(s))ds + \int_0^t S_a(t - s)\sigma(s, x(s))dw(s) \right)
\]
which implies that
\[
E\|x(t)\|^2 \leq 4\|T_a(t)\|^2 \|x_0\|^2 + 4\|T_a(t)\|^2 E\|g(x)\|^2 \\
+ 4 \int_0^t \|S_a(t - s)\|ds \int_0^t \|S_a(t - s)\|E\|f(s, x(s))\|^2 \|H_a\|^2 ds.
\]
Moreover, v(\(\cdot\)) \(\in\) \(L^2\)(\([0, T]\), \(\mathbb{R}\)) and let
\[ x(t) = \int_0^t E(t-s)w(s)ds, \] for all \(t \in [0, T]\), where \(E(t)\) is the infinitesimal generator of an analytic semigroup \((T(t))_{t \geq 0}\) in \(E\). Furthermore, \(A\) has a discrete spectrum with eigenvalues of the form \(-n^2\), \(n \in \mathbb{N}\) and corresponding normalized eigenfunctions are given by \(x_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)\). In addition \([x_n : n \in \mathbb{N}]\) is an orthonormal basis for \(E\).

\[ T(t)y = \sum_{n=1}^{\infty} e^{-n^2t} (y, x_n)x_n, \quad \text{for all } y \in E, \text{ and every } t > 0. \]

From these expressions it follows that \((T(t))_{t \geq 0}\) is a uniformly bounded compact semigroup, so that \(R(\lambda, A) = (\lambda - A)^{-1}\) is a compact operator for all \(\lambda \in \rho(A)\) i.e. \(A \in \mathbb{K}^{\infty}(\theta_0, \omega_0)\).

To represent the above fractional system (4.3) into the abstract form of (4.1), we introduce the functions \(f : J \times \mathbb{R}^n \to \mathbb{R}^n\), \(\sigma : J \times \mathbb{R}^n \to \mathbb{L}^2\) and \(g : \mathbb{R}^n \to \mathbb{R}^n\) by \(f(t, z)(x) = h(t, z(x))\), \(\sigma(t, z)(x) = \psi(t, z(x))\) and \(g(w) = \sum_{i=0}^{p} K w(t_i)\), where \(K(z)(x) = \int_0^z K(x, y)z(y)dy\). Thus, \(f, \sigma\) and \(g\) satisfy the assumption of Theorem 4.4. Hence, by Theorem 4.4 the system (4.3) has a mild solution on \([0, T]\).

**Note 4.6.** The differential inclusion system is considered as a generalization of the system described by differential equations. Ning and Liu [37] discussed the existence of mild solutions of a class of impulsive neutral stochastic evolution
inclusions in Hilbert space in the case where the right hand side is convex or nonconvex-valued by using fixed point theorems for multivalued mappings and evolution system theory. The existence of solutions of nonlinear neutral stochastic differential inclusions with infinite delay in a Hilbert space has been reported in [38]. Upon making some appropriate assumption on functions $f$ and $\sigma$, one can establish the approximate controllability of fractional stochastic differential inclusions with nonlocal conditions by adapting the techniques and ideas established in this paper and suitably introducing the technique of single valued maps defined in [39].

References


