Fractional variational iteration method and its application

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Abstract

Fractional differential equations have been investigated by variational iteration method. However, the previous works avoid the term of fractional derivative and handle them as a restricted variation. We propose herein a fractional variational iteration method with modified Riemann–Liouville derivative which is more efficient to solve the fractional differential equations.

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1. Introduction

The variational iteration method [1–4] has been extensively worked out for many years by numerous authors. Starting from the pioneer ideas of the Inokuti–Sekine–Mura method, Ji-Huan He [3] developed the variational iteration method (VIM) in (1999). In this method, the equations are initially approximated with possible unknowns. A correction functional is established by the general Lagrange multiplier which can be identified optimally via the variational theory. The method provides rapidly the convergent successive approximations of the exact solution. Besides, the VIM has no restrictions or unrealistic assumptions such as linearization or small parameters that are used in the nonlinear operators. The method has matured into a fully fledged theory thanks to the effects of many researchers, notably, A.M. Wazwaz [5], G.E. Draganescu [6,7], M.A. Abdou and A.A. Soliman [8–11], Z. Odibat and S. Momani [12,13] and a detail review can be found in [14].

Recently, a new modified Riemann–Liouville left derivative is proposed by G. Jumarie [15]. Comparing with the classical Caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Secondly, αth derivative of a constant is zero. For these merits, G. Jumarie’s derivative was successfully applied in the probability calculus [16], fractional Laplace problems [17,18] and fractional variational calculus [19]. With the Jumarie’s fractional derivative, we propose a fractional functional and extend VIM to fractional differential equations. It is also applied to solve time fractional and space fractional diffusion equations.

2. Modified Riemann–Liouville derivative

Assume \( f: R \rightarrow R, \, x \rightarrow f(x) \) denote a continuous (but not necessarily differentiable) function and let the partition \( h > 0 \) in the interval \([0, 1] \). Through the fractional Riemann–Liouville integral

\[
\alpha I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) \, d\xi, \quad \alpha > 0,
\]

the modified Riemann–Liouville derivative is defined as

\[
\alpha D_0^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha} (f(\xi) - f(0)) \, d\xi,
\]

where \( x \in [0, 1], \, n-1 \leq \alpha < n \) and \( n \geq 1 \).

G. Jumarie’s derivative is defined through the fractional difference

\[
\Delta_0^\alpha = (FW - 1)^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{x!}{(x-k)!} f(x + (\alpha - k)h),
\]

where \( FW f(x) = f(x + h) \). Then the fractional derivative is defined as the following limit,

\[
f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta_0^\alpha f(x)}{h^\alpha}.
\]

The proposed modified Riemann–Liouville derivative as shown in Eq. (2) is strictly equivalent to Eq. (4). Meanwhile, we would introduce some properties of the fractional modified Riemann–Liouville derivative in Eqs. (5) and (6).
(a) Fractional Leibniz product law
\[ 0D^\alpha_x (uv) = u^{(\alpha)}v + uv^{(\alpha)}. \] (5)

(b) Fractional Leibniz formulation
\[ 0D^\alpha_x f(x) = f(x) - f(0), \quad 0 < \alpha \leq 1. \] (6)

Therefore, the integration by part can be used during the fractional calculus
\[ 0D^\alpha_x u^{(\alpha)}v = (uv)^{\alpha} - 0D^\alpha_x uv^{(\alpha)}. \] (7)

(c) Integration with respect to \((dx)^\alpha\).

Assume \( f(x) \) denote a continuous \( R \to R \) function. We use the following equality for the integral w.r.t. \((dx)^\alpha\) [18],
\[ 0D^\alpha_x f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} f(\xi) \, d\xi = \frac{1}{\Gamma(\alpha + 1)} \int_0^x f(\xi) (d\xi)^\alpha, \quad 0 < \alpha \leq 1. \] (8)

Firstly, let’s revisit VIM for the differential equation
\[ Lu + Nu = g(x), \] (9)
where \( L \) and \( N \) are respectively the linear and nonlinear operators and \( g(x) \) is the source inhomogeneous term. The VIM [1–4] was introduced by He where a corrected functional for Eq. (9) can be written as
\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda [Lu_n(t) + N\tilde{u}_n(t) - g(t)] \, dt, \] (10)

where \( \lambda \) is a general Lagrangian multiplier, which can be identified optimally via the variational theory, and \( \tilde{u}_n \) is a restricted identification of which \( \tilde{u}_n = 0 \). By this method, it is firstly required to determine the Lagrangian multiplier \( \lambda \) that is identified optimally. The successive approximations \( u_{n+1}(x) \) (with \( n > 0 \)) of the solution \( u(x) \) will be readily obtained upon using the determined Lagrangian multiplier and any selective function \( u_0(x) \). Consequently, the solution is given by
\[ u(x) = \lim_{n \to \infty} u_n(x). \] (11)

For variational iteration method, the key is the identification of the Lagrangian multiplier. Recently, there has been a great deal of interest in fractional diffusion equations. Fractional-order diffusion equations are the generalizations of the classical diffusion equations treating the super-diffusive flow processes. These equations arise in continuous-time random walks [20], modeling of anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon. There are several kinds of fractional diffusion equations.

(1) Space–time fractional diffusion equation
\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta}. \] (12)

This model offers an interpretation to phenomena between the heat equation \( (\alpha = \beta = 1) \) and the wave equation \( (\alpha = \beta = 2) \). Y. Fujita [21,22] considered integro-differential equations which exhibit heat diffusion and wave propagation properties. Oldham and Spanier [23] considered a fractional diffusion equation that contains first-order derivative in space and half-order derivative in time.

(2) Space fractional diffusion equation
\[ \frac{\partial u(x,t)}{\partial t} = c(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta}, \quad 0 < \beta \leq 2. \] (13)

In space fractional diffusion process, G. Gorenflo and F. Mainardi [24] obtained this fractional model by replacing the second-order space derivative with a suitable fractional derivative operator. C. Tadjeran et al. [25] presented a second-order accurate numerical approximation for the fractional diffusion equation.

(3) Time fractional diffusion equation
\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c(x) \frac{\partial^2 u(x,t)}{\partial x^2}, \quad \alpha > 0. \] (14)

For standard diffusion \( \alpha = 1 \), whereas in anomalous sub-diffusion \( \alpha < 1 \), and in anomalous super-diffusion \( \alpha > 1 \), this equation can also describe fractional heat-like phenomenon in porous materials.


In this study, we use the modified Riemann–Liouville derivative and propose a fractional variational iteration method. It is illustrated by the following 2 examples.

**Example 1.** As the first example, for simplicity, we consider the following fractional diffusion equation
\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{x^2}{2} \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha \leq 1, \] (15)
which subject to the boundary conditions
\[ u(0,t) = 0 \quad \text{and} \quad u(1,t) = f(t), \] (16)
and the initial condition \( u(x,0) = x^2 \). Then a corrected functional for Eq. (14) can be constructed as follows
\[ u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t',t) \left\{ \frac{\partial^\alpha u_n(x,t')}{\partial t^\alpha} \right\} (dt')^\alpha, \] (17)
with the property from Eq. (7), \( \lambda(t,t) \) must satisfy
\[ \frac{\partial^\alpha \lambda(t,t)}{\partial t^\alpha} = 0 \quad \text{and} \quad 1 + \lambda(t,t)|_{t=t_0} = 0. \] (18)

Therefore, \( \lambda(t,t) \) can be identified as \( \lambda(t,t) = -1 \). Substituting the initial value \( u_0(x,t) = u_0(x,0) = x^2 \) into the iteration formulation as follows
\[ u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x,t')}{\partial t^\alpha} - \frac{x^2}{2} \frac{\partial^2 u_n(x,t')}{\partial x^2} \right\} (dt')^\alpha. \] (19)

We can derive
\( u_1(x, t) = x^2 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \tau)}{\partial \tau^\alpha} - \frac{x^2 \partial^2 u_0(x, \tau)}{2} \right\} (d\tau)^\alpha \),
\( = x^2 + \frac{1}{\Gamma(1+\alpha)} \int_0^t x^2 (d\tau)^\alpha = x^2 + \frac{x^2 t^\alpha}{\Gamma(1+\alpha)} \). \tag{20}

By the same manipulation, we have
\( u_2(x, t) = x^2 + \frac{x^2 t^\alpha}{\Gamma(1+\alpha)} + \frac{x^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \),
\( u_3(x, t) = x^2 + \frac{x^2 t^\alpha}{\Gamma(1+\alpha)} + \frac{x^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^2 t^{3\alpha}}{\Gamma(1+3\alpha)} \). \tag{21}

Generally, we find
\( u_n(x, t) = \sum_{k=0}^n \frac{x^2 t^{k\alpha}}{\Gamma(1+k\alpha)} \), \tag{22}
whose limit has a compact form as
\( u(x, t) = \lim_{n \to \infty} u_n(x, t) = \lim_{n \to \infty} \sum_{k=0}^n \frac{x^2 t^{k\alpha}}{\Gamma(1+k\alpha)} = x^2 E_\alpha(t^\alpha). \tag{23} \)

With the property \( 0D_\alpha^\beta E_\alpha(t^\alpha) = E_\alpha(t^{\beta+\alpha}) \), we can readily check \( u(x, t) = x^2 E_\alpha(t^\alpha) \) is an exact solution of Eq. (15).

Example 2. In order to illustrate the efficiency of our method, we replace the fractional-order \( \beta \ (0 < \beta \leq 2) \) by the order \( 2\gamma \ (0 < \gamma \leq 1) \) in Eq. (13)
\[ \frac{\partial u(x, t)}{\partial t} = -\frac{\partial^{2\gamma} u(x, t)}{\partial x^{2\gamma}}, \quad 0 \leq x \leq 1, \tag{24} \]
where \( \frac{\partial^{2\gamma}}{\partial x^{2\gamma}} \) is defined by \( 0D_\alpha^{2\gamma} \), subject to the boundary conditions
\[ u(0, t) = e^t \quad \text{and} \quad u(1, t) = e^t \cos \gamma(1), \tag{25} \]
and the initial condition \( u(x, 0) = \cos \gamma(x^2) \) where \( u(0, 0) = \cos \gamma(0) \) is generalized cosine function defined by Mittag-function
\( E_\gamma(x^2) = \sum_{k=0}^\infty \frac{x^{2k}}{\Gamma(k+\gamma)} \) and \( \cos \gamma(x) = \frac{E_\gamma(x^2) + E_\gamma(-x^2)}{2} \).

Construct the following functional
\[ u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ u_n^{(2\gamma)} + \frac{\partial u_n(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma. \tag{26} \]

Direct calculation leads to
\[ \delta u_{n+1} = \delta u_n + \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ u_n^{(2\gamma)} + \frac{\partial u_n(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma \]
\[ = \delta u_n + \lambda \delta u_n^{(\gamma)}|_{\xi=x} - \lambda^{(\gamma)} \delta u_n|_{\xi=x} + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda^{(2\gamma)} \delta u_n (d\xi)^\gamma. \tag{27} \]

Similarly, set the coefficients of \( \delta u_n \) to zero
\[ 1 - \lambda^{(\gamma)}|_{\xi=x} = 0 \quad \text{and} \quad \lambda^{(2\gamma)} = 0. \tag{28} \]

The generalized Lagrange multiplier can be identified by the above equations,
\[ \lambda(x, \xi) = \frac{(\xi-x)^\gamma}{\Gamma(1+\gamma)}. \tag{29} \]

Substituting Eq. (29) into the functional Eq. (26) yields the iteration formulation as follows
\[ u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(1+\gamma)} \int_0^t \left\{ u_n^{(2\gamma)} + \frac{\partial u_n(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma. \tag{30} \]

From the initial value \( u_0(x, t) = u(0, t) = e^t \), we can derive
\[ u_1(x, t) = u_0(x, t) + \frac{1}{\Gamma(1+\gamma)} \int_0^t \left\{ u_0^{(2\gamma)} + \frac{\partial u_0(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma \]
\[ = e^t + \frac{e^t \Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \int_0^t \left\{ u_0^{(2\gamma)} + \frac{\partial u_0(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma \]
\[ = e^t - e^t \frac{x^2}{\Gamma(1+2\gamma)}. \tag{31} \]
\[ u_2(x, t) = u_1(x, t) + \frac{1}{\Gamma(1+\gamma)} \int_0^t \left\{ u_1^{(2\gamma)} + \frac{\partial u_1(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma \]
\[ = e^t - e^t \frac{x^2}{\Gamma(1+2\gamma)} + e^t \frac{x^2}{\Gamma(1+4\gamma)} \int_0^t \left\{ u_1^{(2\gamma)} + \frac{\partial u_1(x, \xi, t)}{\partial t} \right\} (d\xi)^\gamma \]
\[ = e^t - e^t \frac{x^2}{\Gamma(1+2\gamma)} + e^t \frac{x^4}{\Gamma(1+4\gamma)} - e^t \frac{x^6}{\Gamma(1+6\gamma)}. \tag{32} \]

More generally,
\[ u_n(x, t) = \sum_{k=0}^n e^t \frac{(-1)^k x^{2k\gamma}}{\Gamma(1+2k\gamma)}. \tag{31} \]

We can have a compact form
\[ u(x, t) = \lim_{n \to \infty} u_n(x, t) = \lim_{n \to \infty} \sum_{k=0}^n e^t \frac{(-1)^k x^{2k\gamma}}{\Gamma(1+2k\gamma)} \]
\[ = e^t \cos \gamma(x^2). \tag{32} \]
which is the exact solution of Eq. (24) and compatible with the conditions in Eq. (25).

3. Conclusion

Variational Iteration Method has proven as an efficient tool to solve nonlinear differential equations of integer order. In this Letter, fractional variational iteration method is given. Now fractional differential equations with modified Riemann–Liouville derivative can be solved by this technique.
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References